JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

NSC FINLAND

Flat-band superconductivity

Academy of Finland
Research funding and expertise

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Some of the codes available at
https://gitlab.jyu.fi/jyucmt/

Collaborators

Teemu's thesis: https://r.jyu.fi/teemuthesis

Risto's thesis: https://r.jyu.fi/ristothesis

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Contents

- What I mean by flat bands
- Connection between superconducting order parameter and electronic dispersion

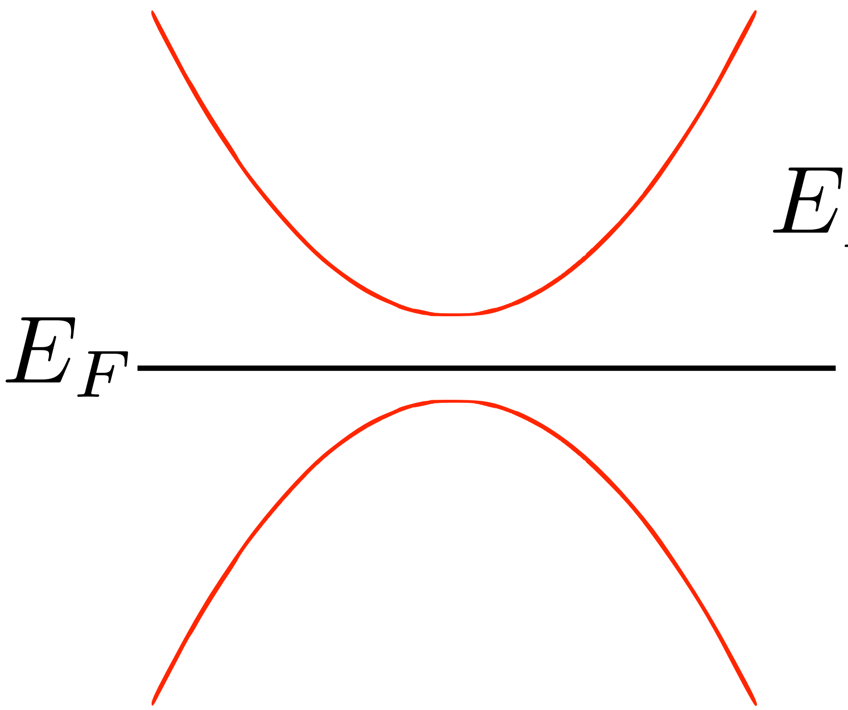
$$T_c \sim \Delta \sim g\Omega_{\mathrm{FB}}$$

Attractive interaction   Flat-band size

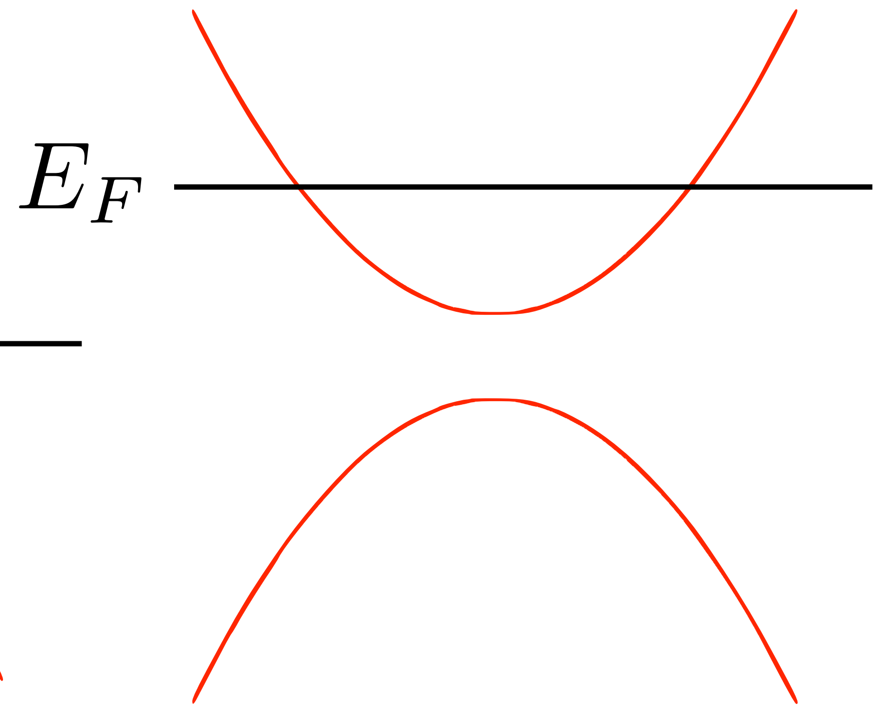
- Graphene-based examples
- Electron-phonon model (Eliashberg)

Not much "unconventional", but will discuss a degeneracy between s- and f-wave superconductivity

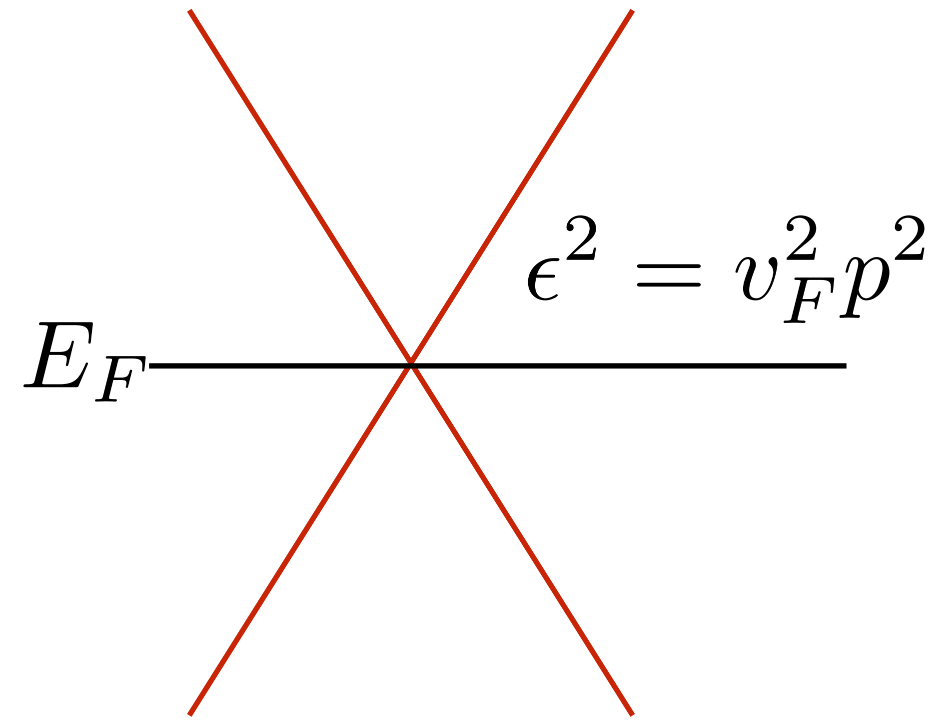
Classifying electron spectra



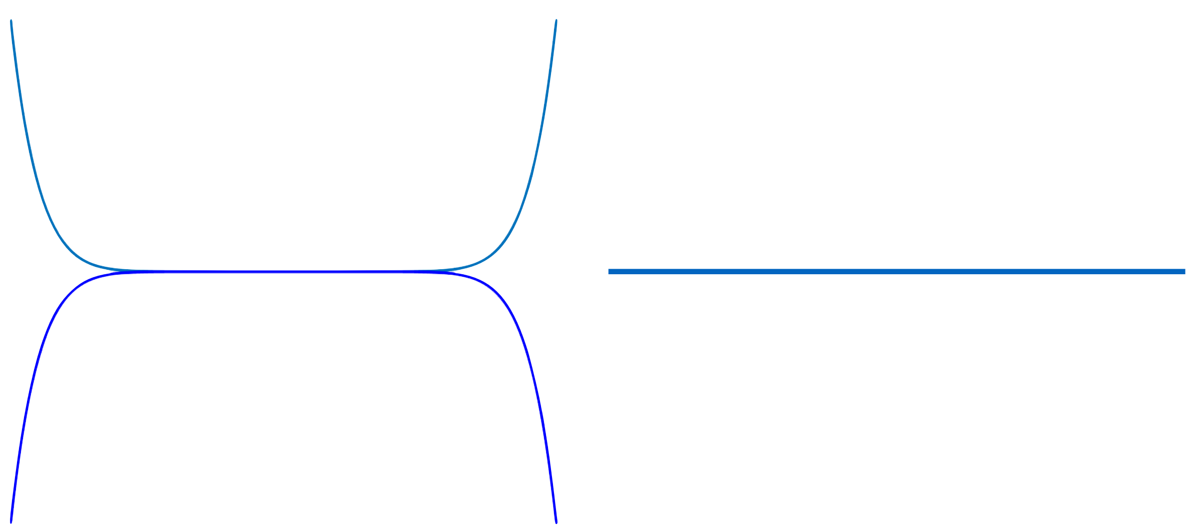
Band insulator: boring



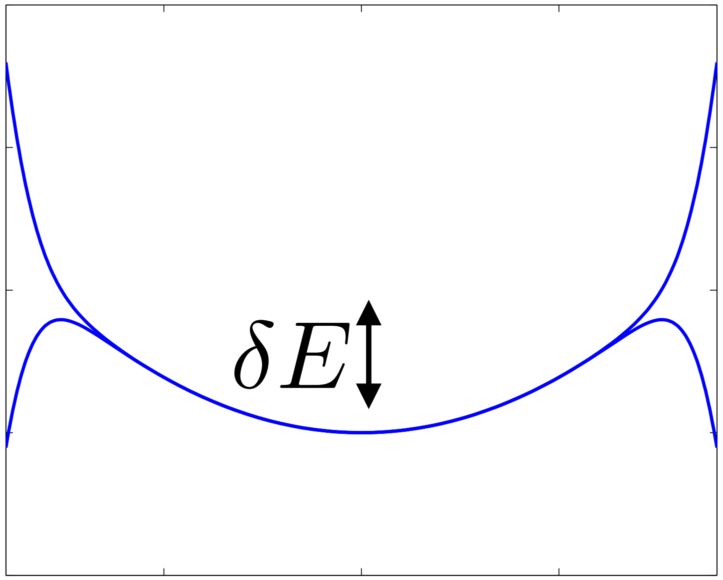
Metal: Fermi surface physics



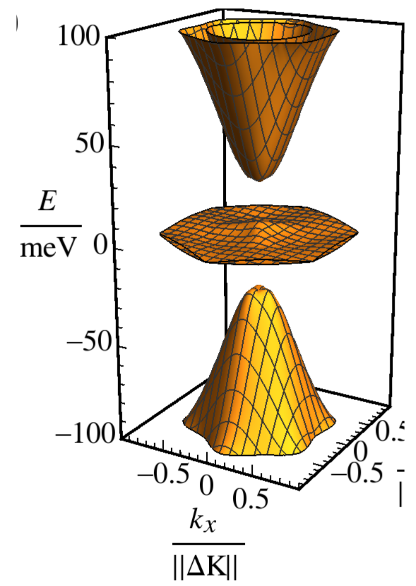
Semimetal: Dirac/Weyl point



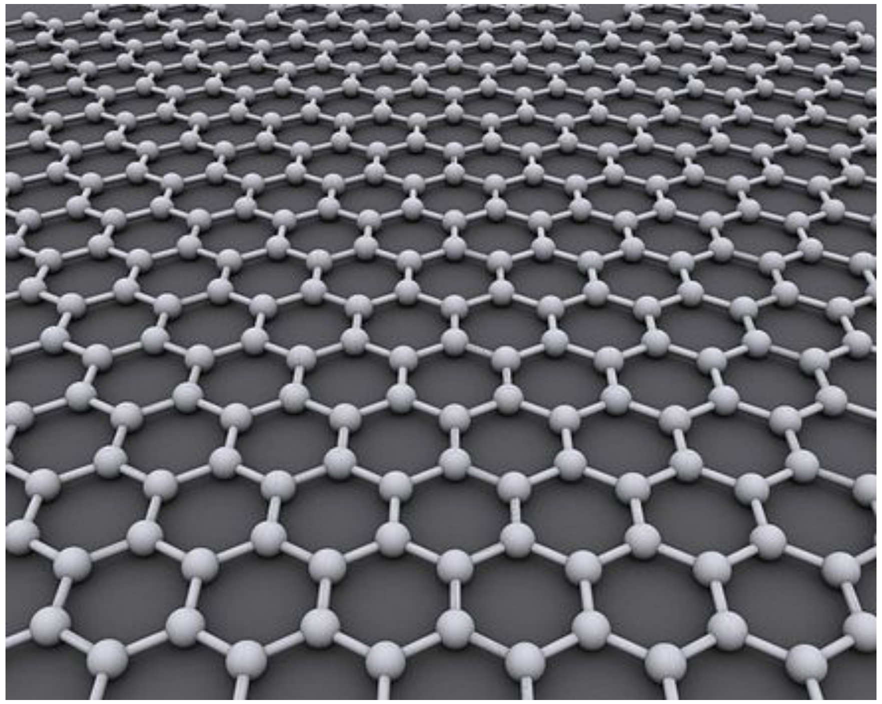
"Ideal" flat band

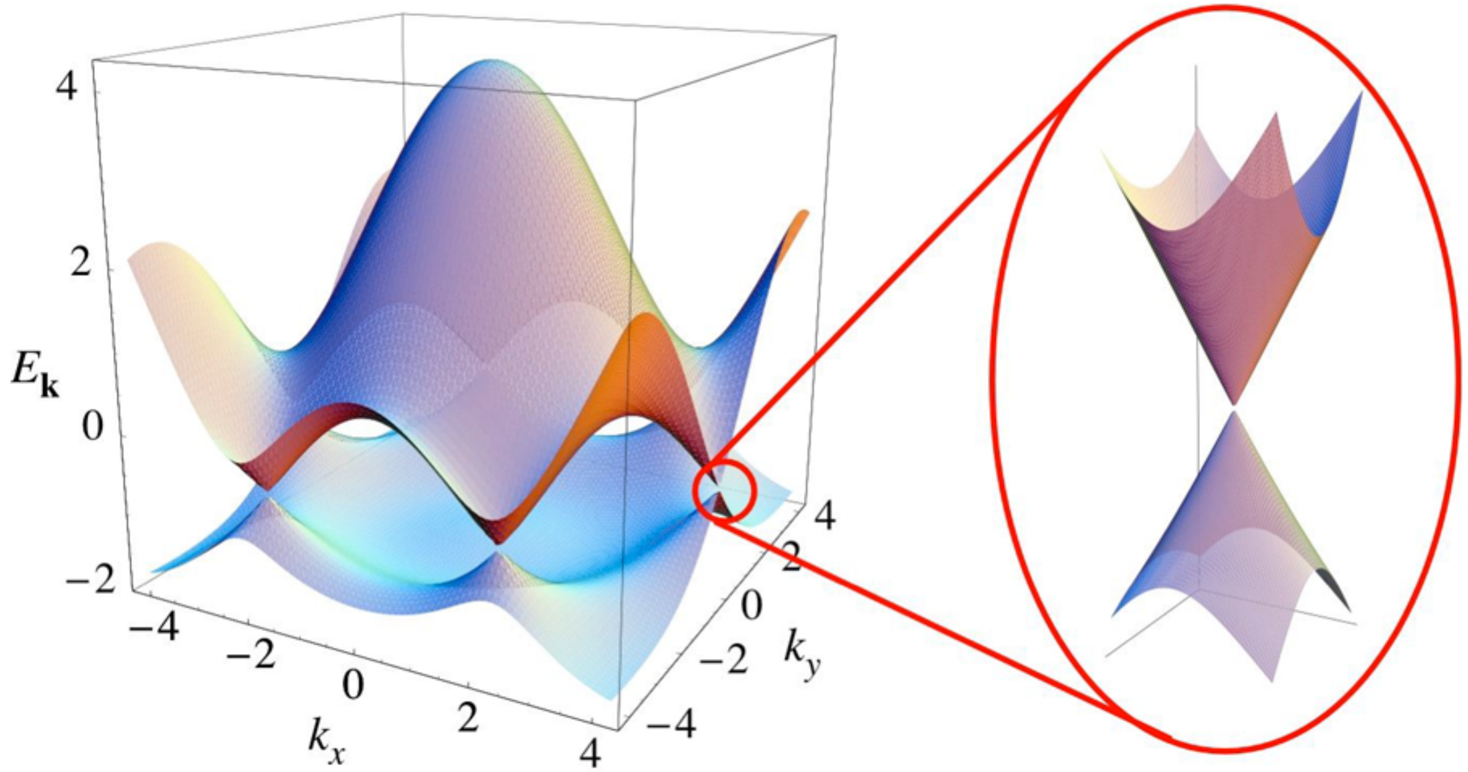


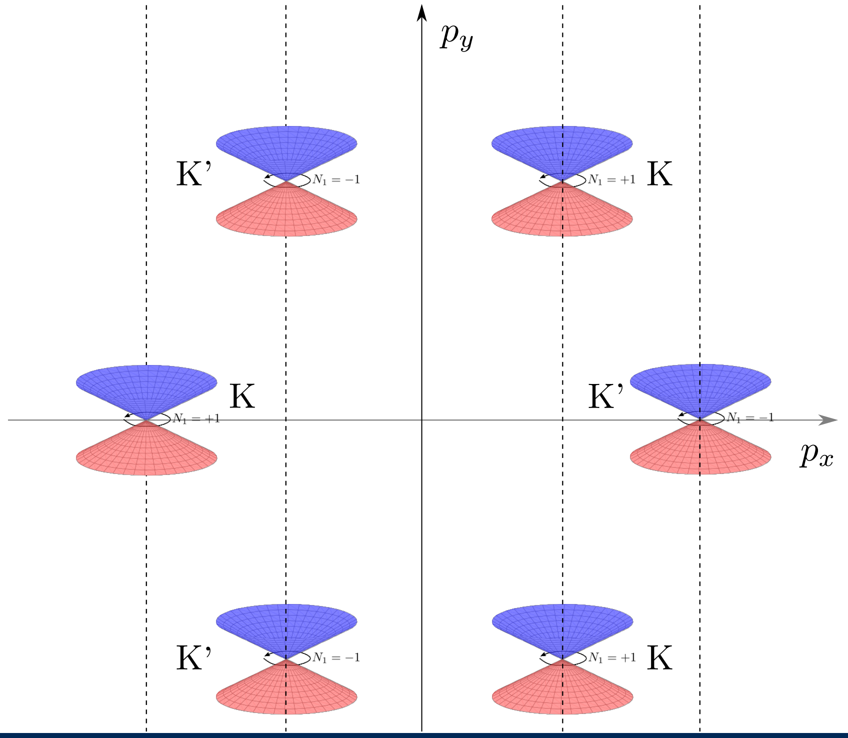
Flat band with a finite bandwidth



Example semimetal: graphene





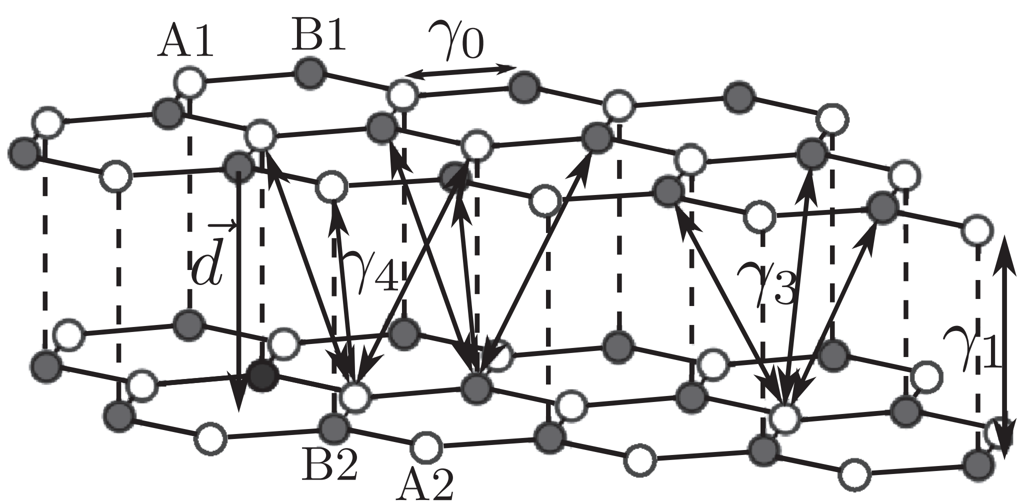


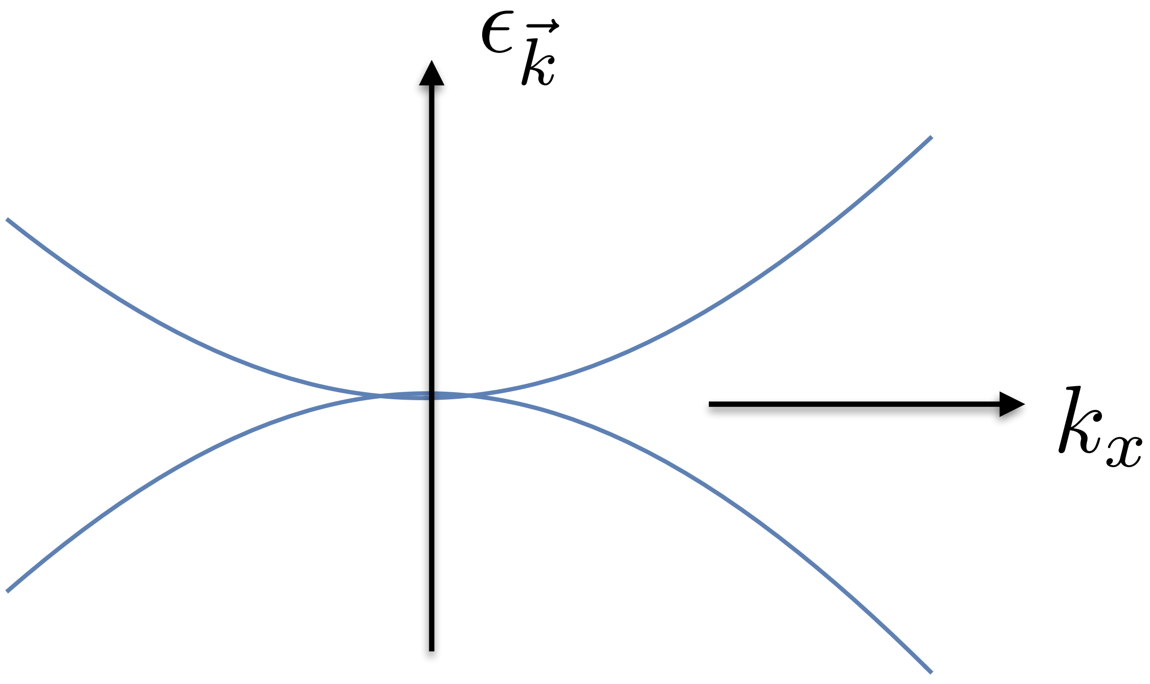
Relevant (and related) concepts:

- Band touching
- Sublattice “spinor” structure
- Two valleys

(Simple) bilayer graphene

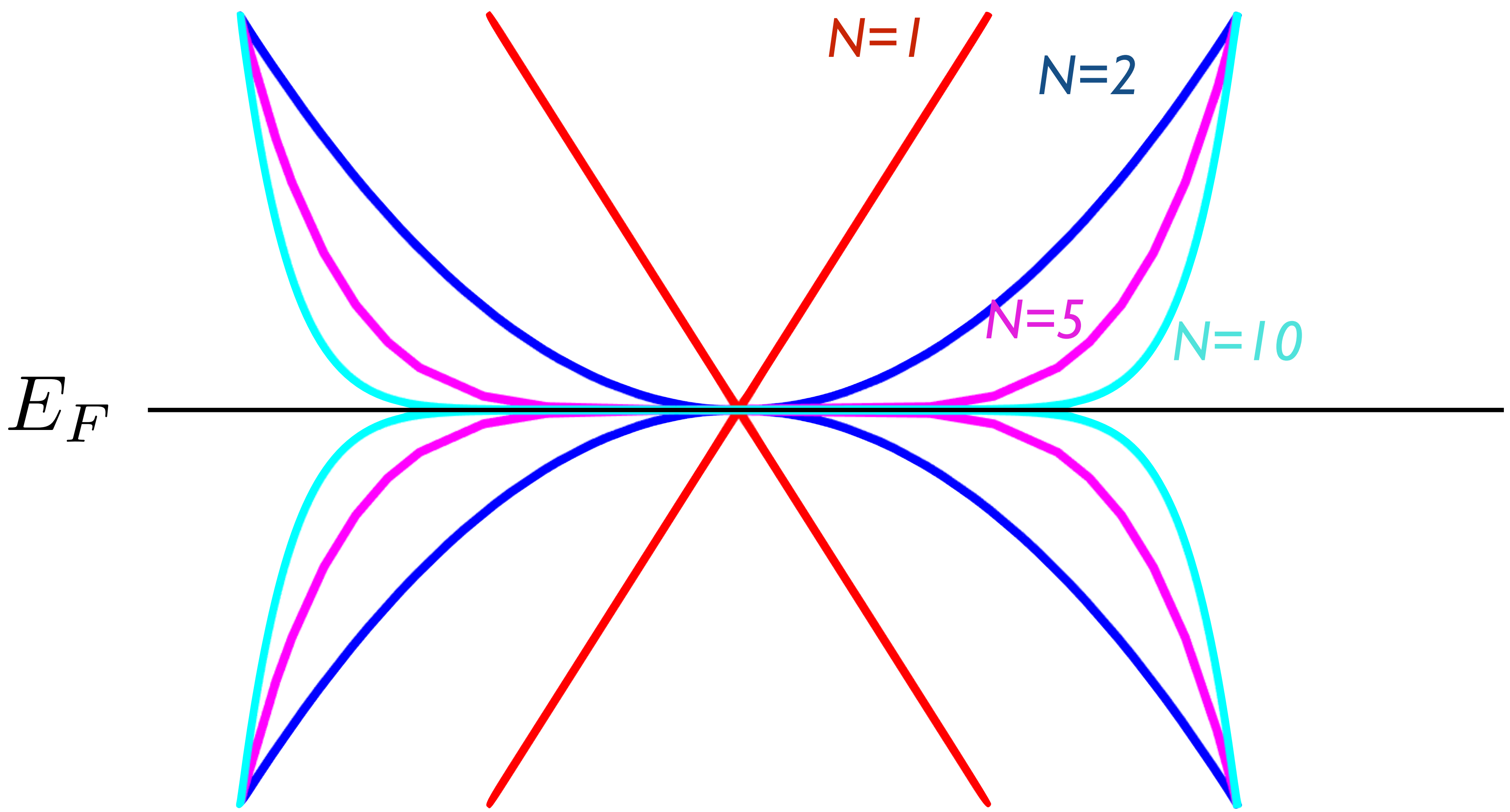
Semimetal with quadratic touching!



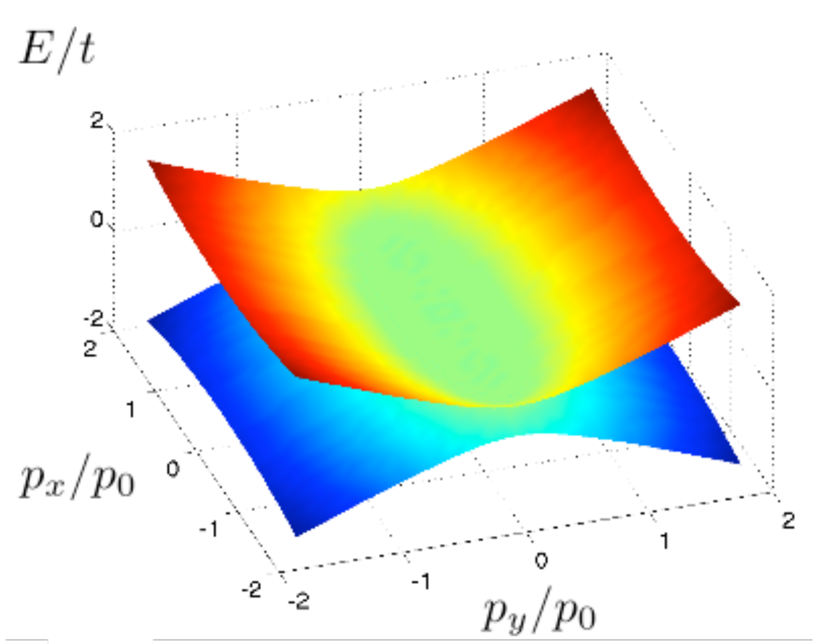


Towards flat bands

Rhombohedral multilayer



$$\epsilon^2 \propto p^{2N}$$



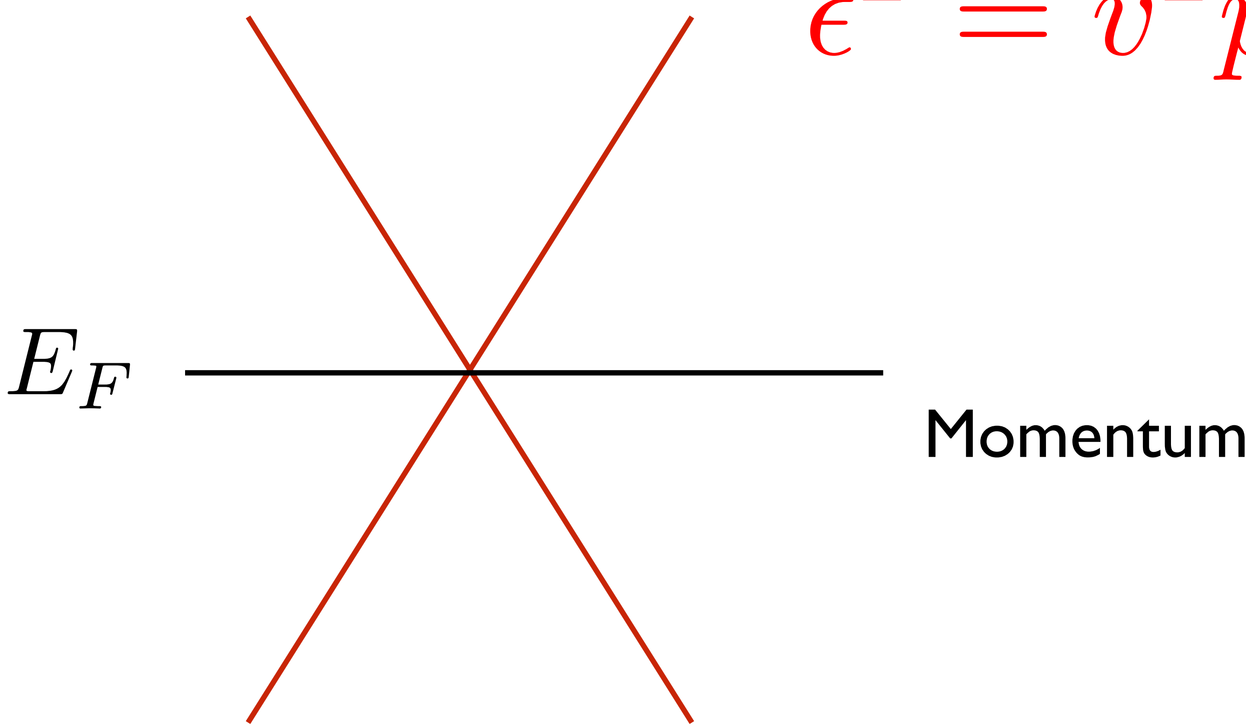
Different kinds of semimetal dispersions: asymptotically towards flat bands, an almost equi-energy area Ω_{FB} in momentum space

Towards flat bands

(Twisted few-layer graphene and periodically strained graphene)

Alternative:

$$\epsilon^2 = v^2 p^2$$



Flatness depends on a reference scale!

Adding interactions

$$
\begin{aligned}
H &= H^{(1)} + H^{(2)} + \cdots \\
&= \sum_{\mathbf{k},\sigma,n} \epsilon_{\mathbf{k},n,\sigma} c^{\dagger}_{\mathbf{k},n,\sigma} c_{\mathbf{k},n,\sigma} + \sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\sigma,\sigma'} V(\mathbf{q}) c^{\dagger}_{\mathbf{k}+\mathbf{q},\sigma} c^{\dagger}_{\mathbf{k}'-\mathbf{q},\sigma'} c_{\mathbf{k}',\sigma'} c_{\mathbf{k},\sigma} + \cdots
\end{aligned}
$$

"One-body spectrum" "Two-body interactions"

Here, simplification (contact interaction): $V(\mathbf{q}) = g = \text{const.}$.

$$[g] = \text{energy} \cdot \text{length}^d$$

Connection to lattice models

Take, for example, the (attractive) Hubbard model with on-site interaction:

$$H = \sum_{\sigma,i,j} h_{i,j} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_{i} c_{i\uparrow}^{\dagger} c_{i\downarrow}^{\dagger} c_{i\downarrow} c_{i\uparrow}$$

Here: $c_{i\sigma}$ can be a multiorbital vector

$$[U] = \text{energy} \qquad U = g/A_c = g\Omega_{\mathrm{BZ}}/(2\pi)^d$$

Unit cell area

Volume of the 1st Brillouin zone

Correlated states from interactions

$$H = H^{(1)} + H^{(2)} + \cdots$$
$$= \sum_{\mathbf{k},\sigma,n} \epsilon_{\mathbf{k},n,\sigma} c^\dagger_{\mathbf{k},n,\sigma} c_{\mathbf{k},n,\sigma} + \sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\sigma,\sigma'} V(\mathbf{q}) c^\dagger_{\mathbf{k}+\mathbf{q},\sigma} c^\dagger_{\mathbf{k}'-\mathbf{q},\sigma'} c_{\mathbf{k}',\sigma'} c_{\mathbf{k},\sigma} + \cdots$$

"One-body spectrum" "Two-body interactions"

Most commonly used approach: Hartree-Fock mean-field theory

(Typically local interactions, on-site or nearest neighbour)

$|\text{ground state}\rangle \approx$ Fermi gas of a suitably defined operator

$\Rightarrow$ emergence of *mean fields*

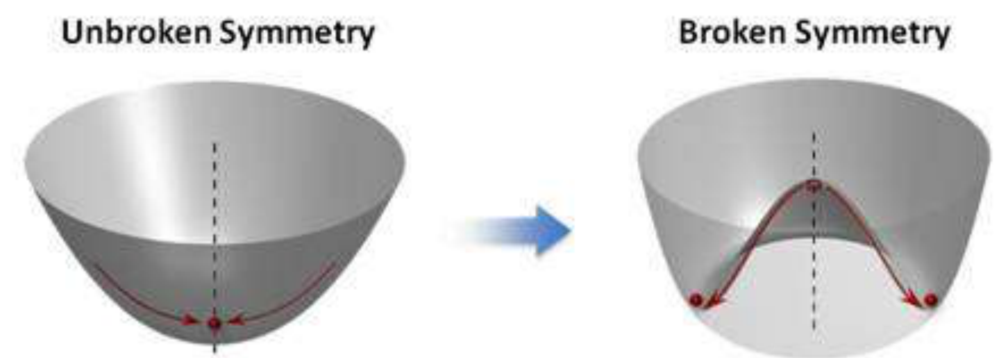
$\Rightarrow$ often associated with spontaneous symmetry breaking

Superconductivity

Example of a broken-symmetry state

Order parameter: $F_{\sigma\sigma'}(\vec{r}) = \langle \Psi_\sigma(\vec{r}) \Psi_{\sigma'}(\vec{r}) \rangle$

$$\text{Supercurrent: } F = |F| e^{i\phi} \Rightarrow \vec{j}_s = D_s \nabla \phi$$



Picture sources: Phys. org, Wikimedia Commons

Superconductivity

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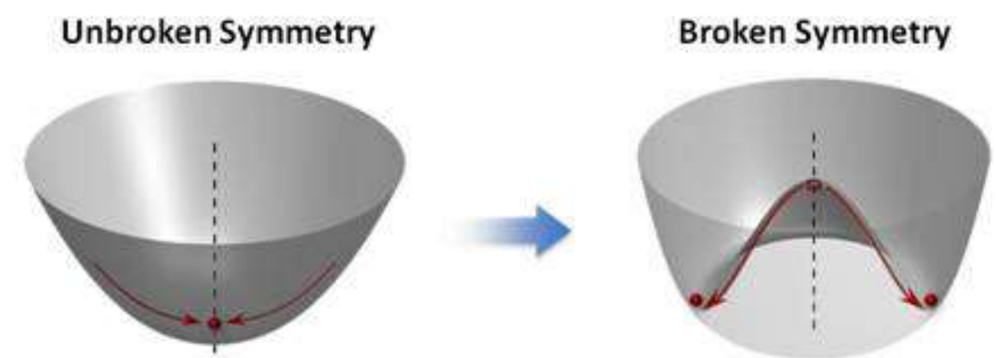

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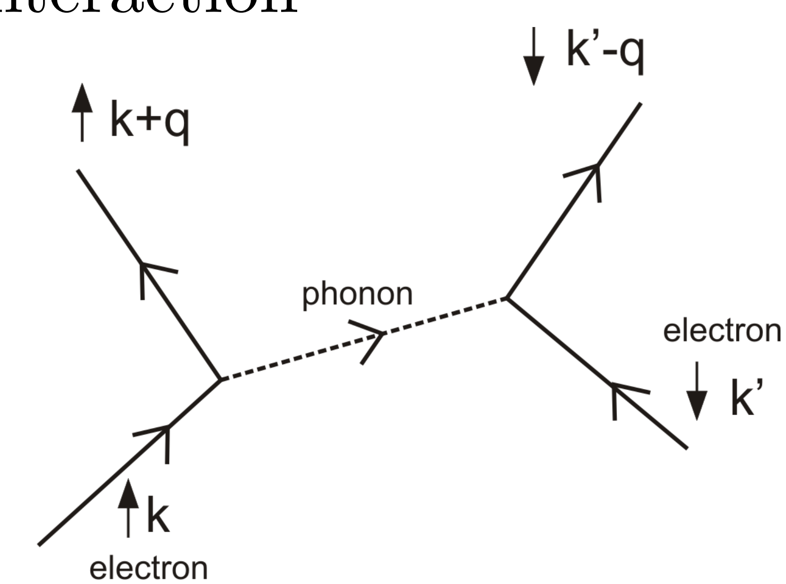
Supercurrent: $F = |F| e^{i\phi} \Rightarrow \vec{j}_s = D_s \nabla \phi$

BCS and Cooper instability: an arbitrarily weak attractive interaction between electrons leads to the superconducting state at $T = 0$

Bardeen, Migdal and Eliashberg (and others): attractive interaction due to electron-phonon coupling

BCS limit: assume (almost) instantaneous interaction



Picture sources: Phys. org, Wikimedia Commons

Mean field theory

Simplest superconducting dispersion: $E(\vec{p}) = \pm\sqrt{\epsilon(\vec{p}) + \Delta^2}$

Free energy density of superconducting state as a function of the pair potential

$$F(\Delta) = \frac{\Delta^2}{g} - \frac{1}{\Omega}\sum_{\vec{p}}[E(\vec{p}) - \epsilon(\vec{p})]$$

Condensation energy Change in kinetic energy

Mean field Δ at $T = 0$ obtained from the minimum of $F(\Delta)$

$$\Delta = \frac{g}{\Omega}\sum_{\vec{p}} \frac{\Delta}{2\sqrt{\Delta^2 + \epsilon^2((\vec{p}))}} = \frac{g}{(2\pi)^d}\int^{p_c} d\vec{p}\frac{\Delta}{2\sqrt{\Delta^2 + \epsilon^2((\vec{p}))}}$$

$T_c \sim \Delta$

Cutoff p_c (or $E_c(p_c)$) (attractive interaction or the dispersion)

Mean field vs dispersion

$$\Delta = \frac{|g|}{2} \int \frac{d^d p}{(2\pi\hbar)^d} \frac{\Delta}{E_{\mathbf{p}}(\Delta)}$$

Fermi surface: $\int d^3p \to \int \nu(\epsilon) d\epsilon \approx \nu(\epsilon_F) \int d\epsilon$

$$\Delta = \epsilon_c e^{-1/(|g|\nu_F)} = 1.764 T_c \qquad \epsilon_c = \min(\omega_D, \delta E)$$

Exp suppression!

Mean field vs dispersion

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Exp suppression!

Linear dispersion: $\epsilon^2 = v_F^2 p^2$ (2D case)

$$\Delta \sim \frac{g^2 - g_c^2}{g g_c^2} \qquad g_c = \pi^2 v_F^2 \hbar^2 / \epsilon_c$$

Quantum critical point!

$g > g_c$

(Kopnin & Sonin, PRL 2009)

Mean field vs dispersion

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(Kopnin & Sonin, PRL 2009)

Flat band: $\Delta \gg \epsilon(\vec{p})$ for $p \in \Omega_{FB}$

$$\Delta \sim g\Omega_{\mathrm{FB}}$$

Linear in g!

Kopnin, TTH, G.E. Volovik, PRB **83**, 220503 (R) (2011)

Mean field vs dispersion

$$\Delta = \frac{|g|}{2}\int \frac{d^d p}{(2\pi\hbar)^d}\frac{\Delta}{E_{\mathbf{p}}(\Delta)}$$

dimension of ν_F: $1/(\text{energy}\cdot\text{length}^d)$

Fermi surface:

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Linear in g!

dimension of Ω_{FB}: $1/(\text{length}^d)$

Kopnin, TTH, G.E. Volovik, PRB **83**, 220503 (R) (2011)

Ok, but is it relevant?

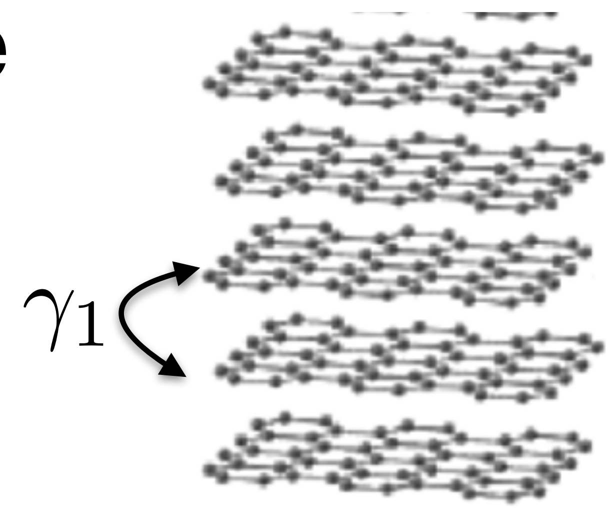
- Where to find flat bands in real systems?
- Can they actually have non-vanishing supercurrent?
- How to describe the source of attractive interaction?
- What about other correlated phases?
- How to maximise the critical temperature?

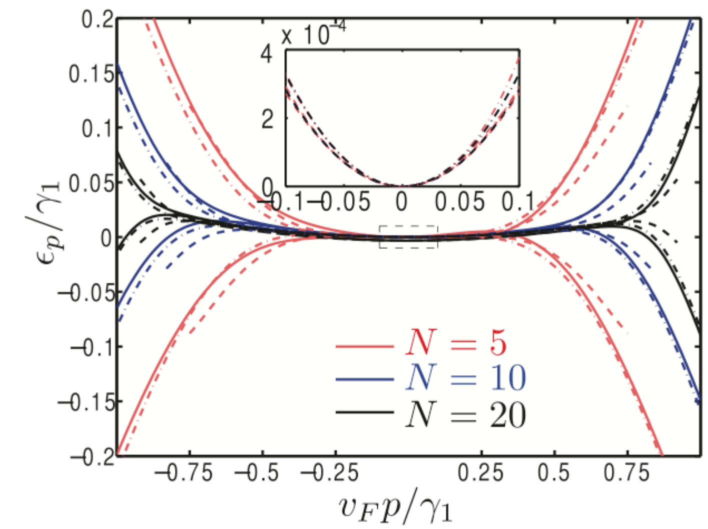
Graphene-based flat bands

- ABC stacked graphene

$$\Omega_{\mathrm{FB}} \sim \pi p_{\mathrm{FB}}^2 = \pi \left(\frac{\gamma_1}{v_F}\right)^2$$

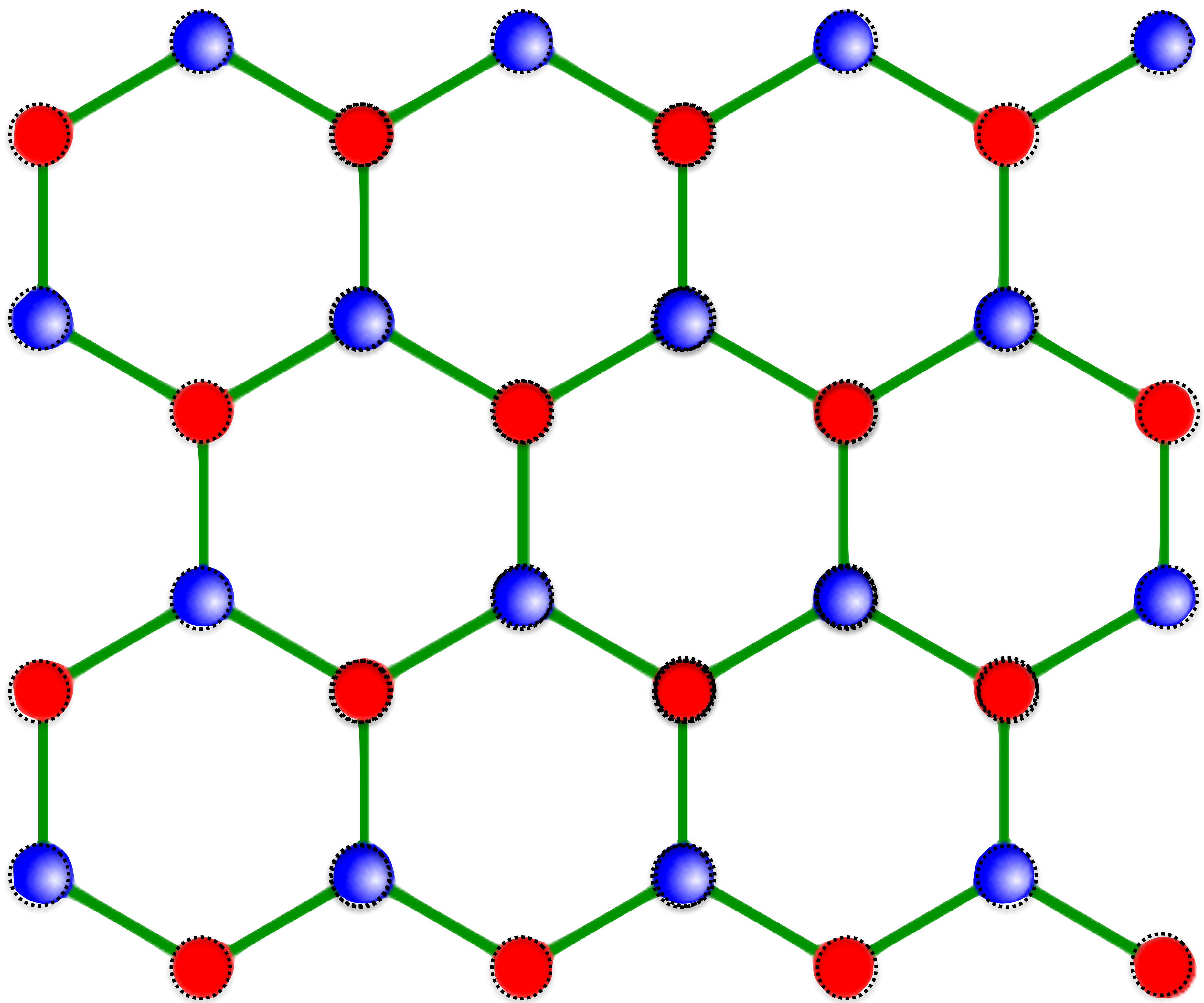
Kopnin, TTH, G.E. Volovik, PRB **83**, 220503 (R) (2011); N.B. Kopnin, M. Ijäs, A. Harju, and TTH, PRB **87**, 140503(R) (2013)





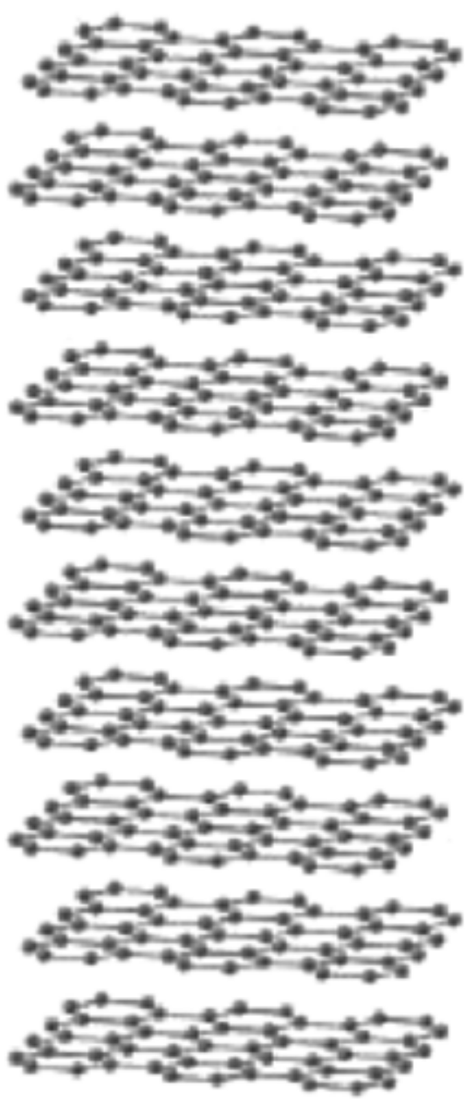
Bernal stacking of pairs of layers

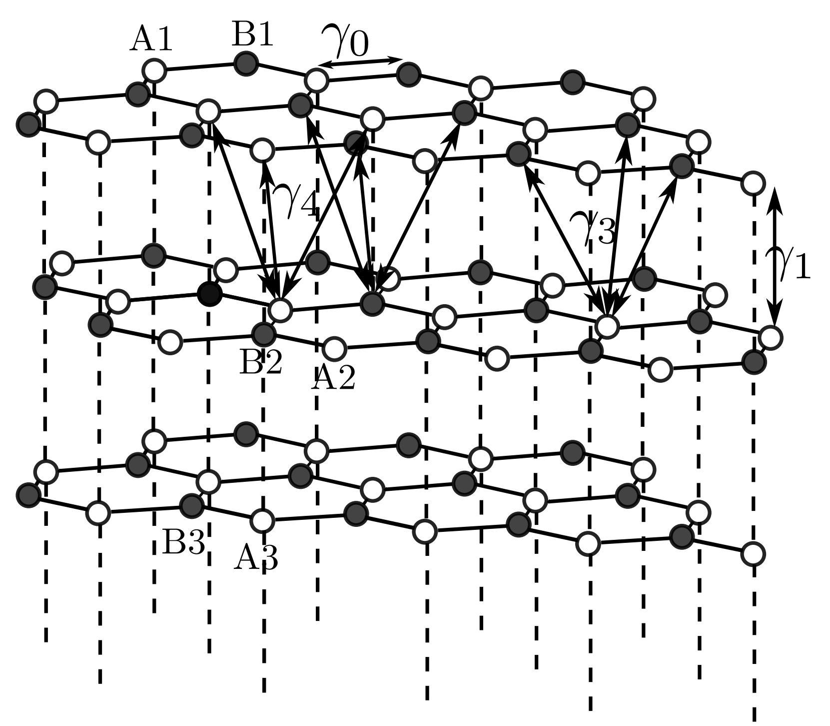
Upper layer:



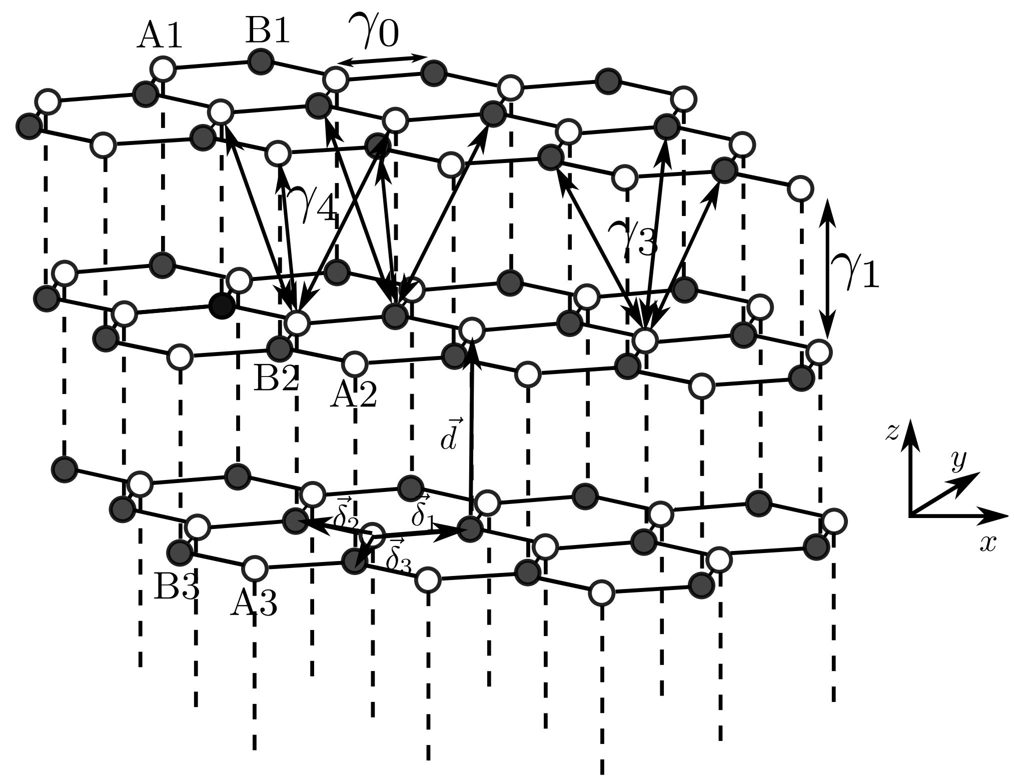
TTH & G.E.Volovik, arXiv:1504.05824

From graphene to graphite

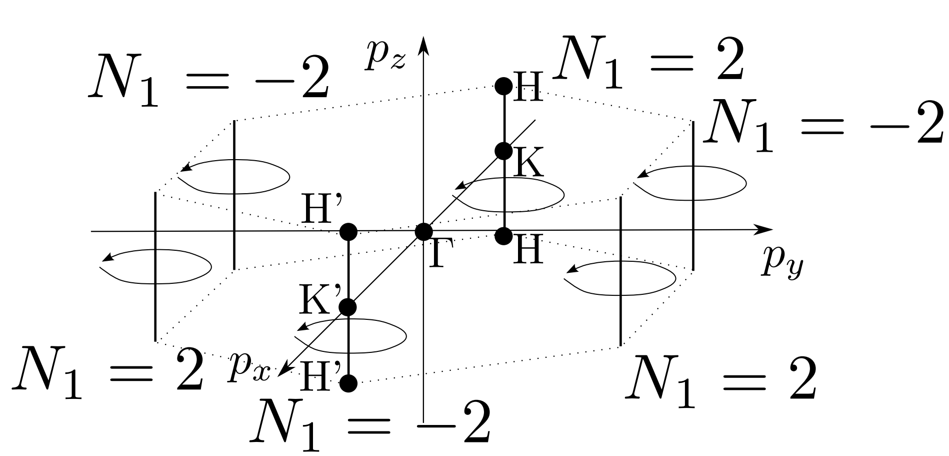


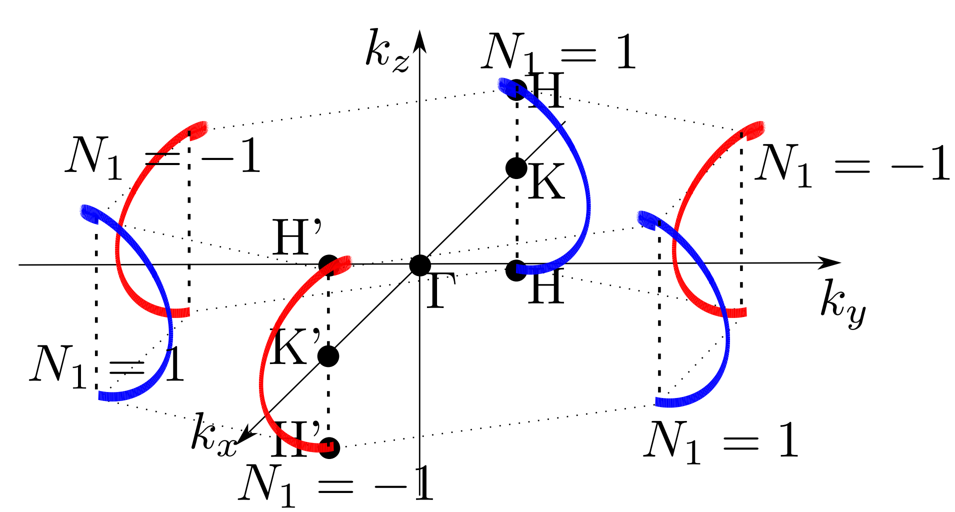


Bernal



Rhombohedral





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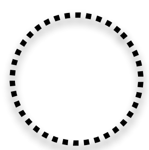

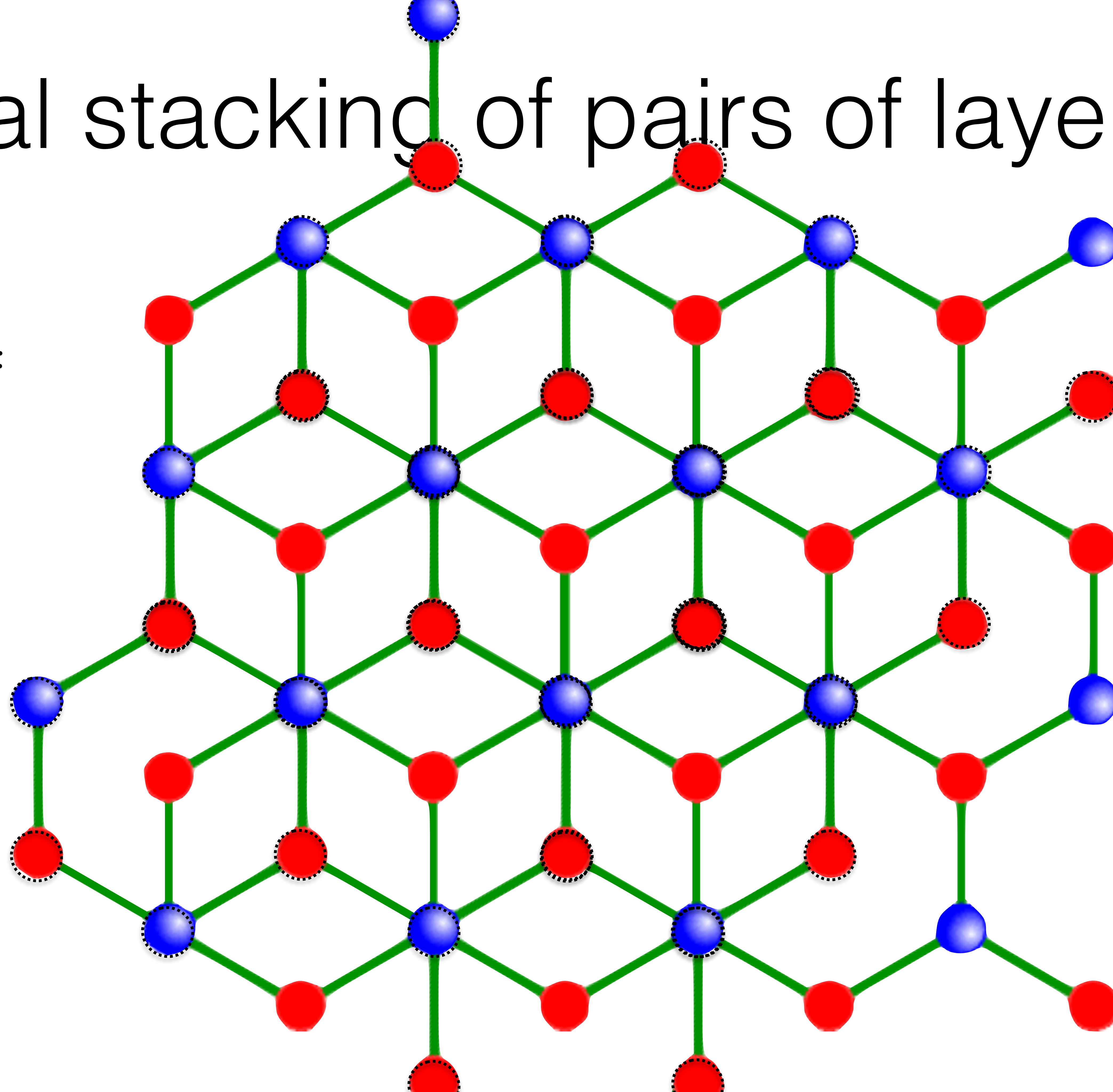

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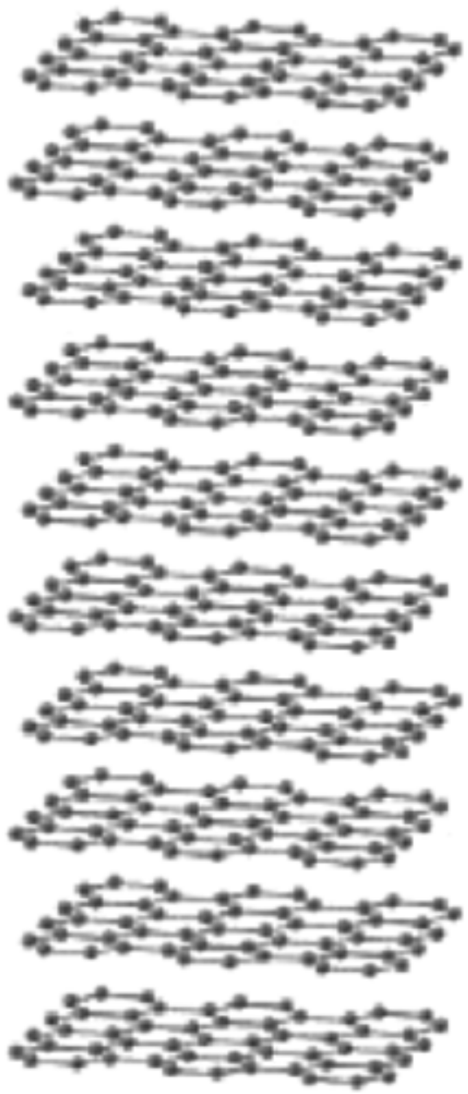
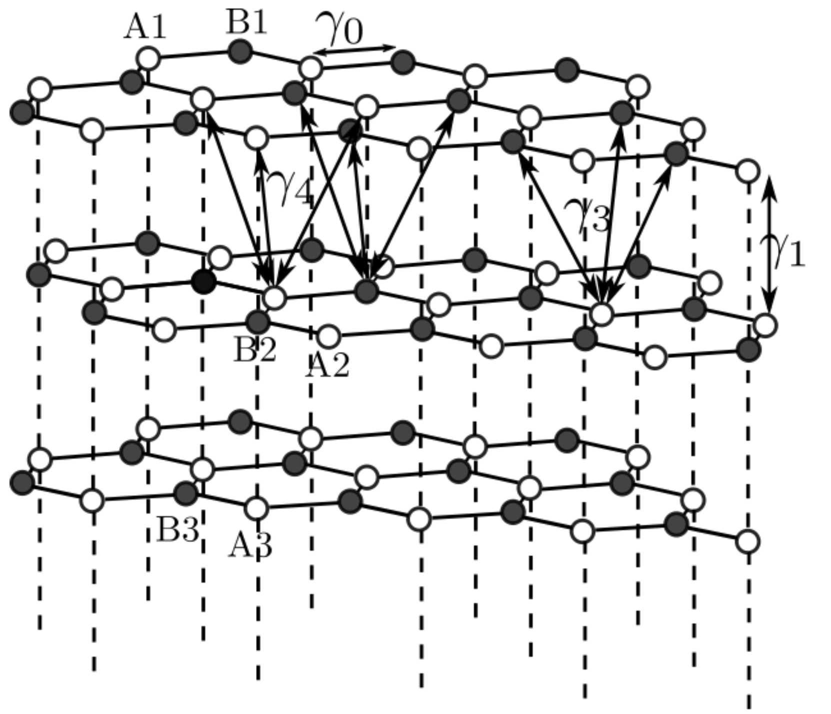
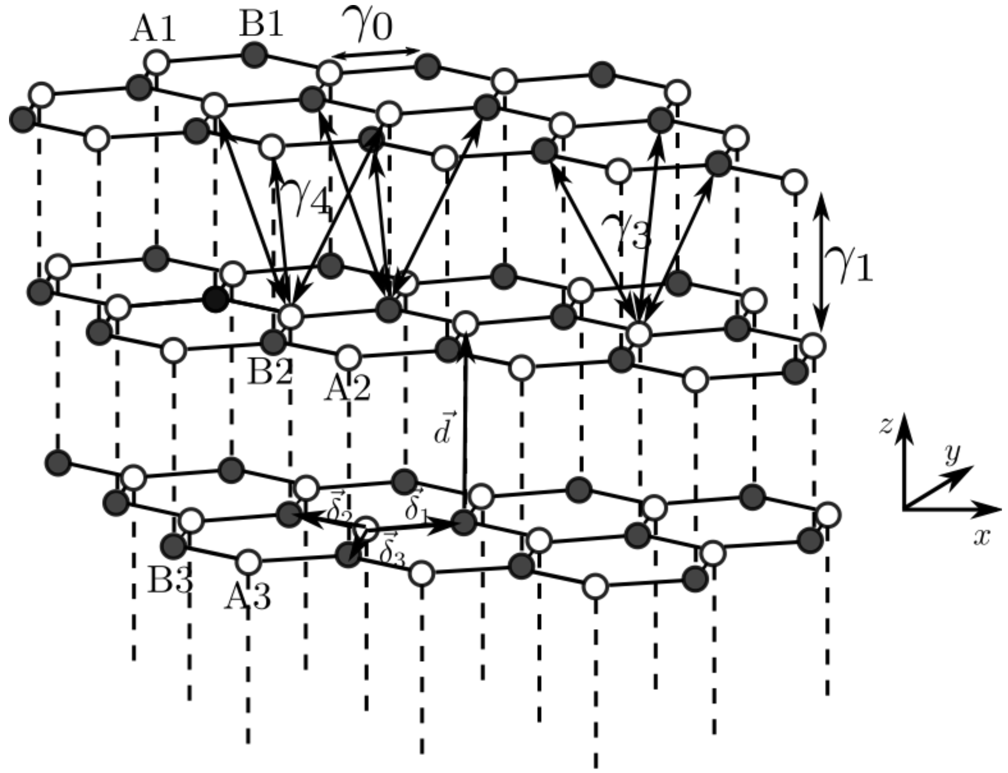
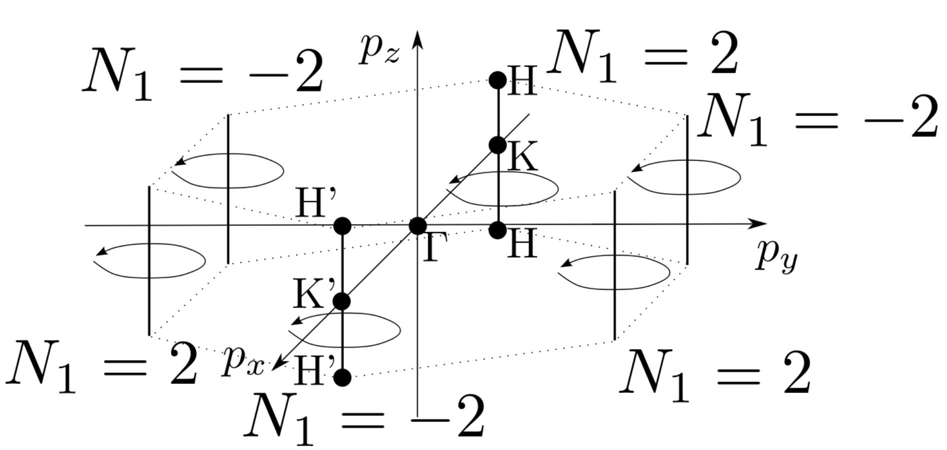


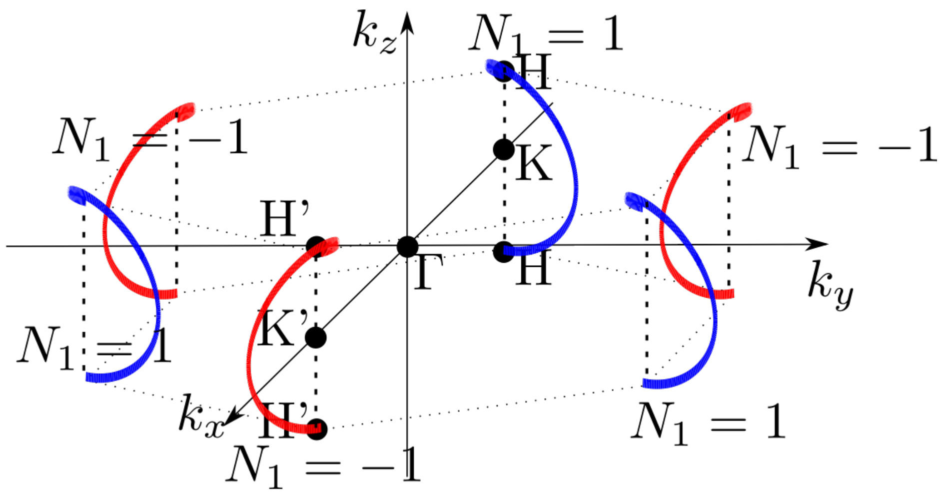


Bernal



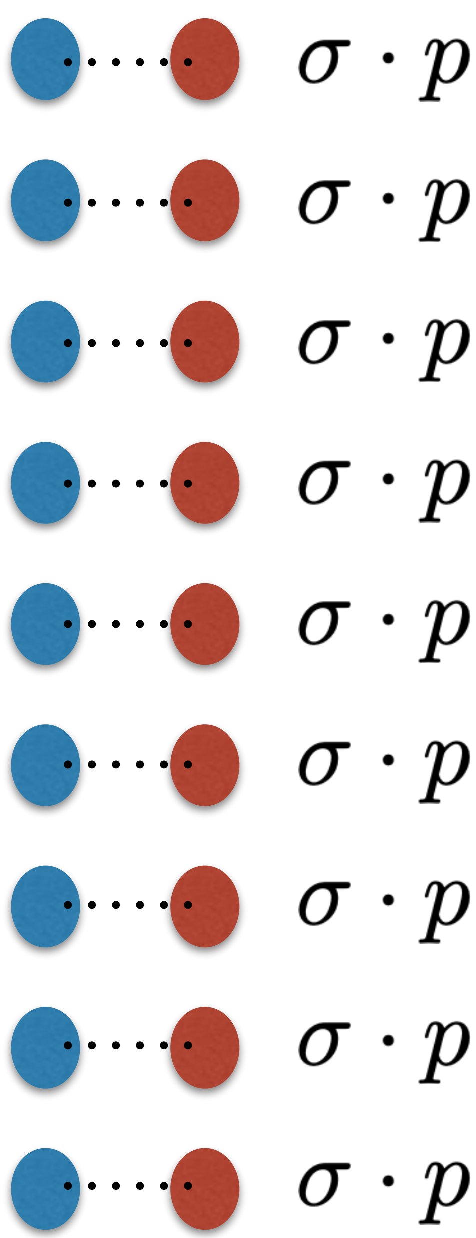
Rhombohedral



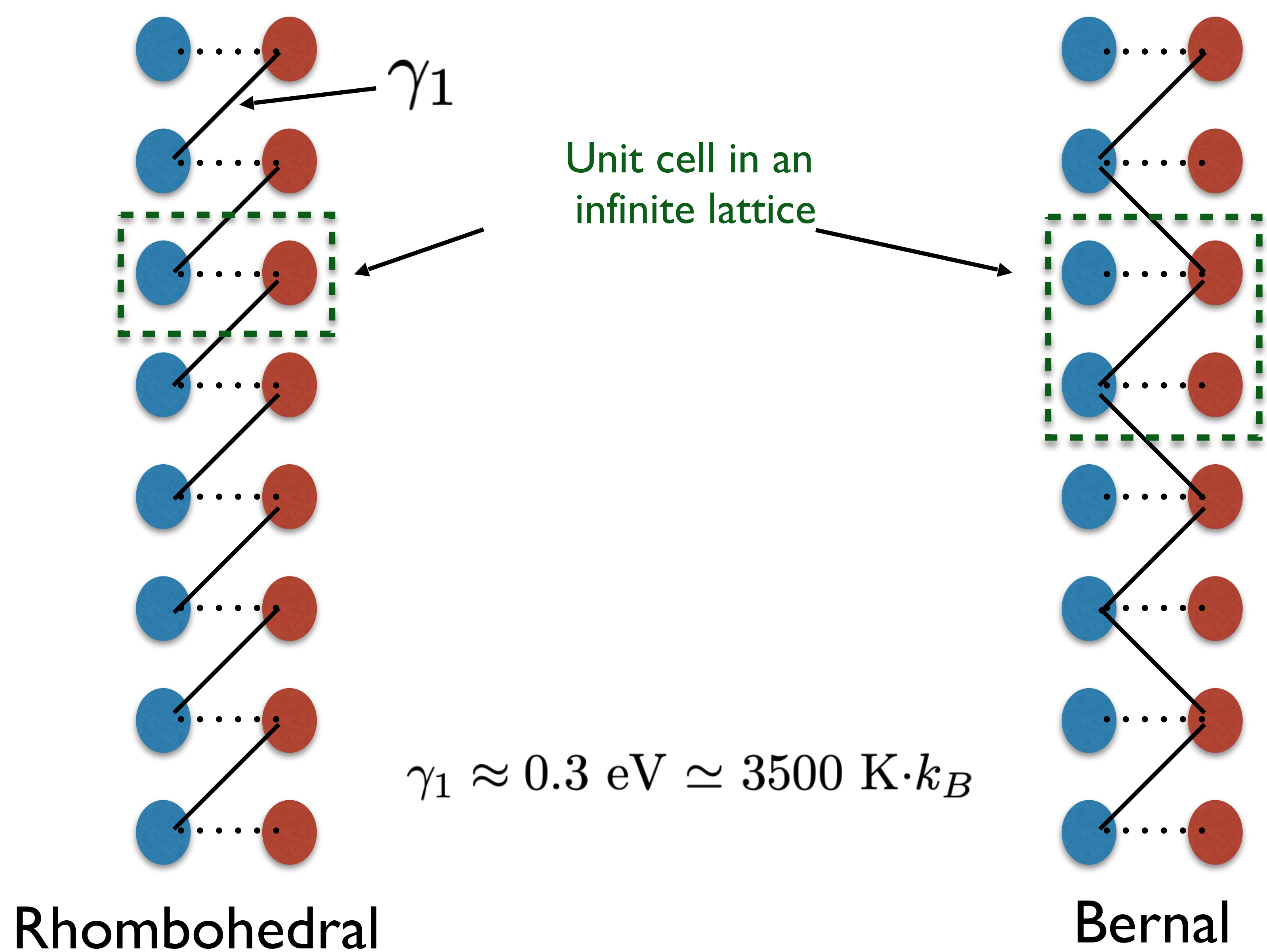


Strongest hopping only

Rule: couple A to B, but not B to A



Two possibilities (and anything in between):



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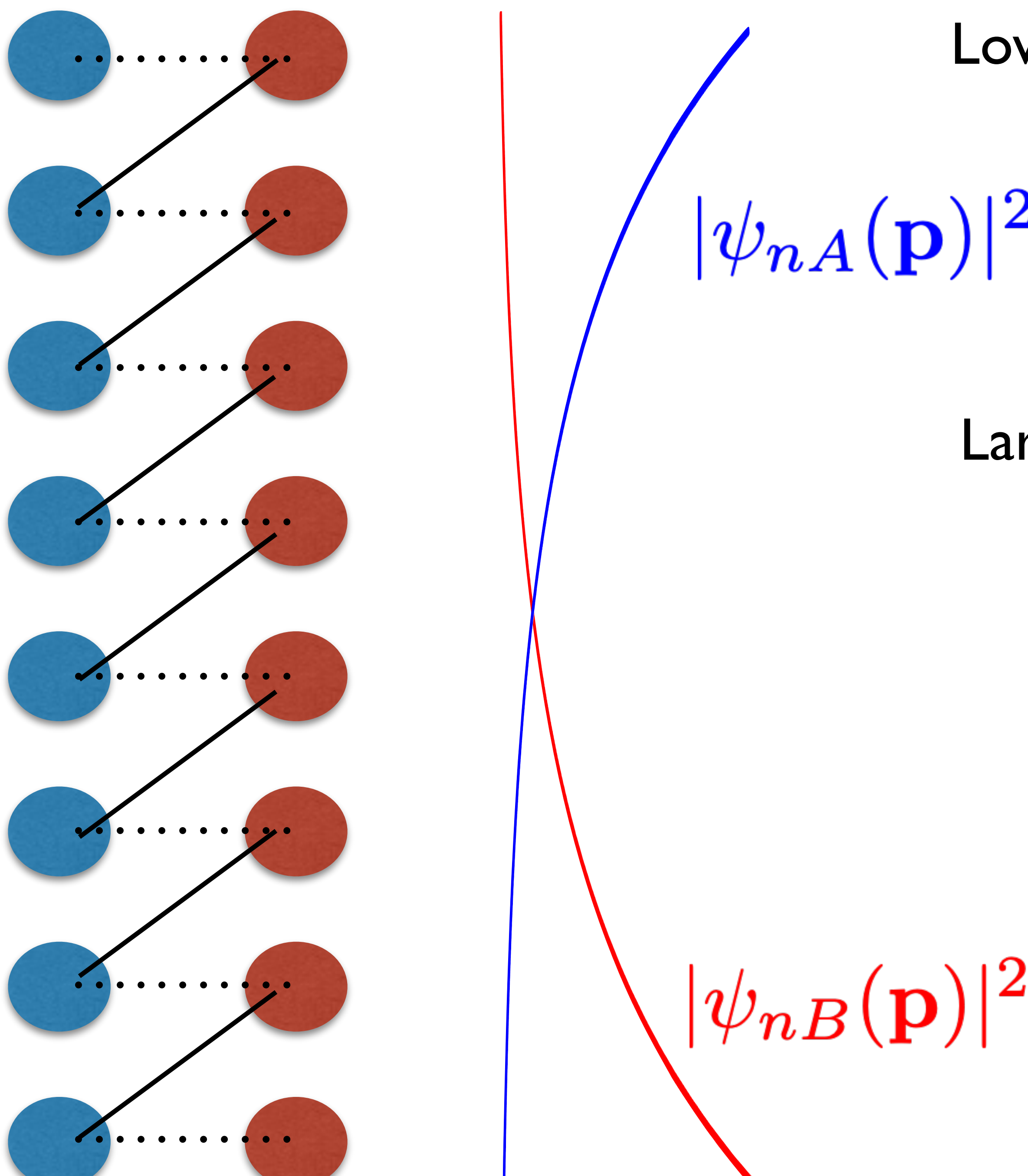
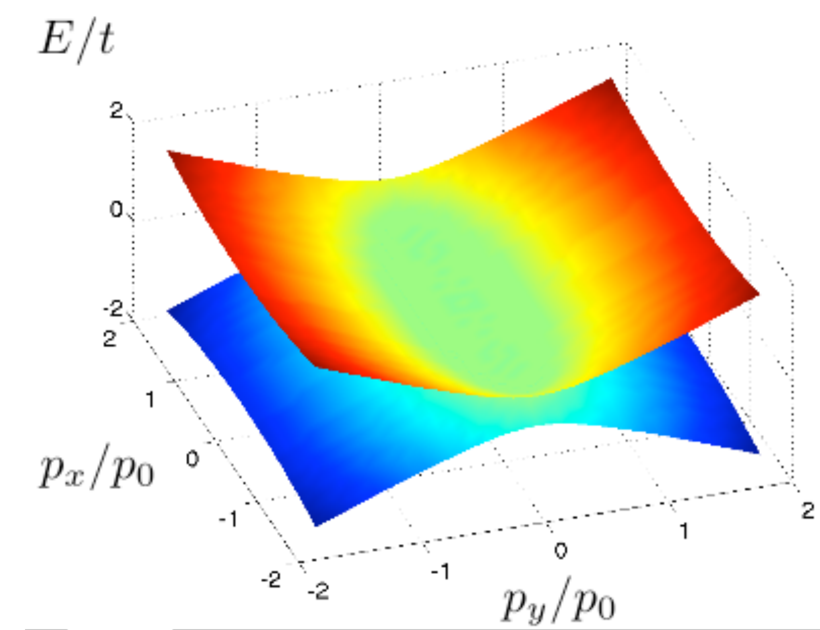


Low-energy spectrum: decaying surface states

$$\epsilon_p \sim \pm p^N$$

Large *N*: flat band!

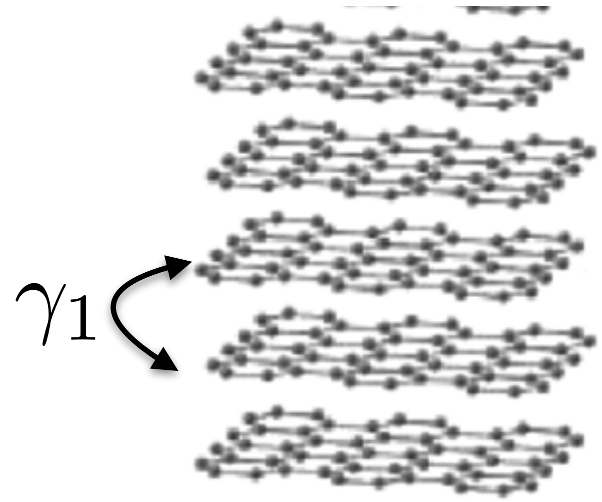


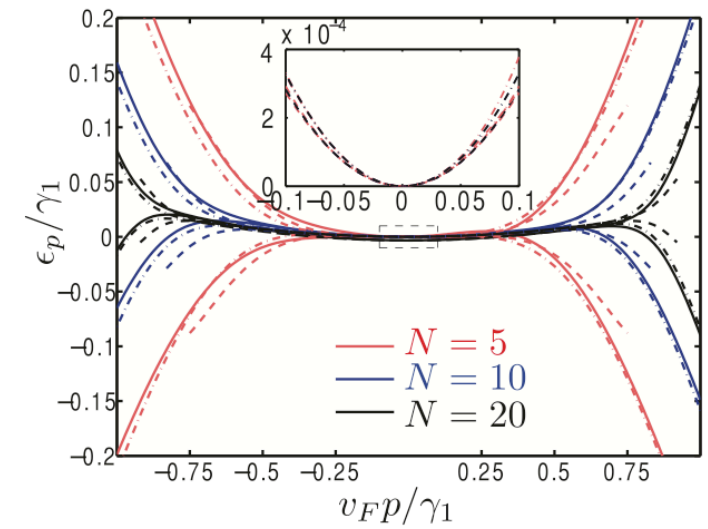
$$\Omega_{\mathrm{FB}} \sim \pi p_{\mathrm{FB}}^2 = \pi \left(\frac{\gamma_1}{v_F}\right)^2$$

Graphene-based flat bands

- ABC stacked graphene

$$\Omega_{\rm FB} \sim \pi p_{\rm FB}^2 = \pi \left(\frac{\gamma_1}{v_F}\right)^2$$





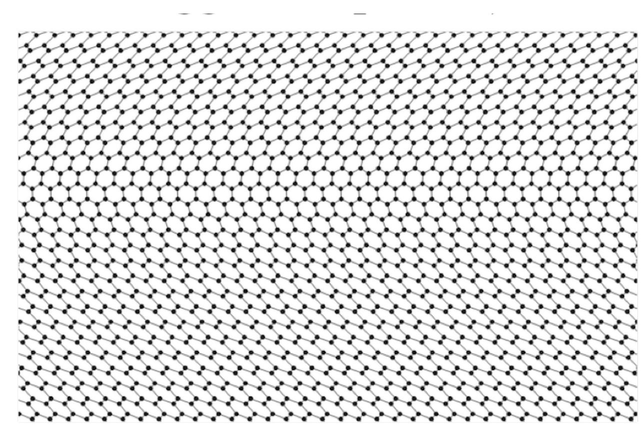
- Periodically strained graphene

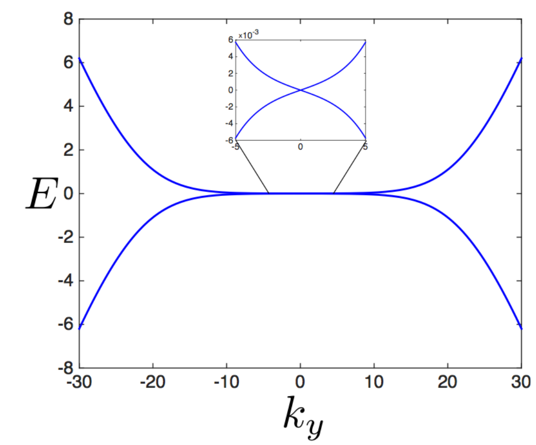
$$\Omega_{\rm FB} \sim \frac{2\pi\beta}{L^2}$$

Strain amplitude

Strain period

TTH & G.E.Volovik, arXiv:1504.05824; Kauppila, Aikebaier, TTH, PRB 2016; Peltonen & TTH, JPCM 2020





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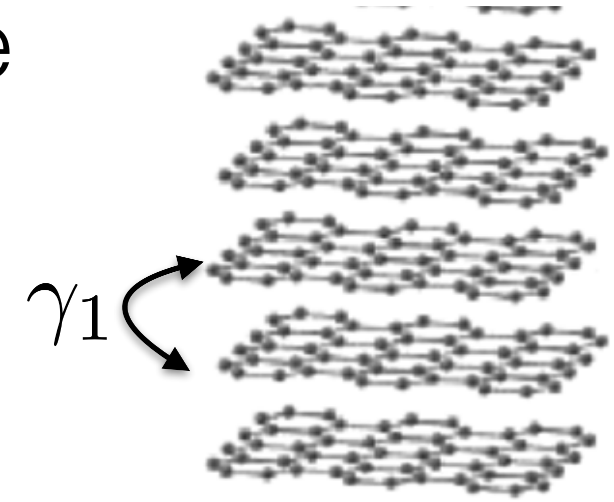

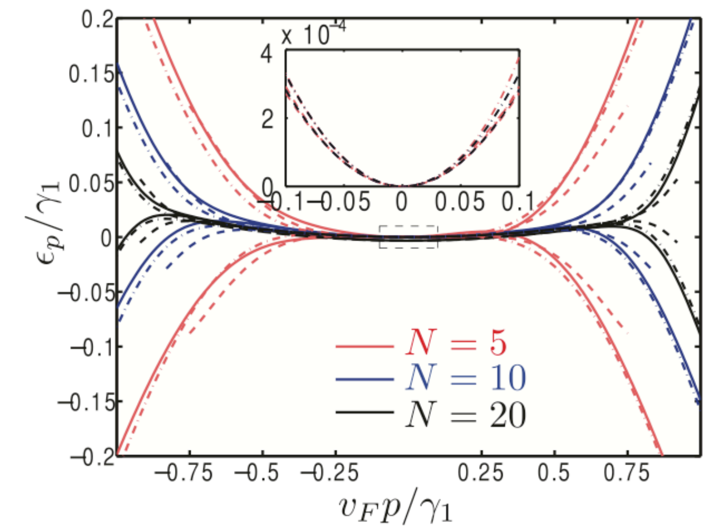

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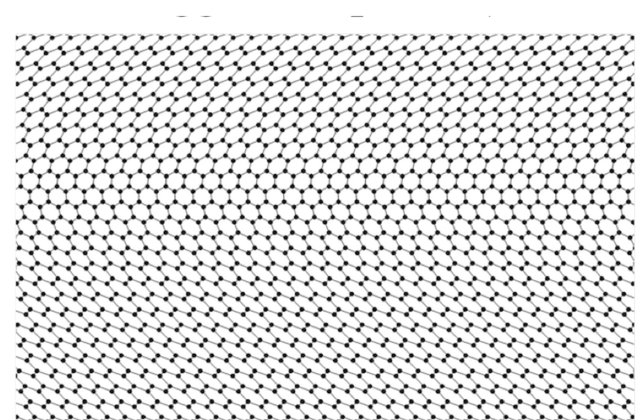
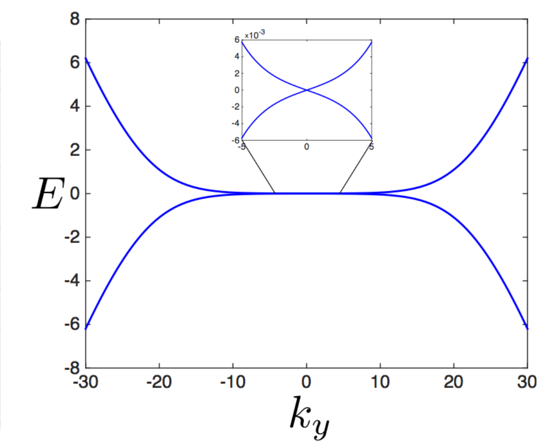


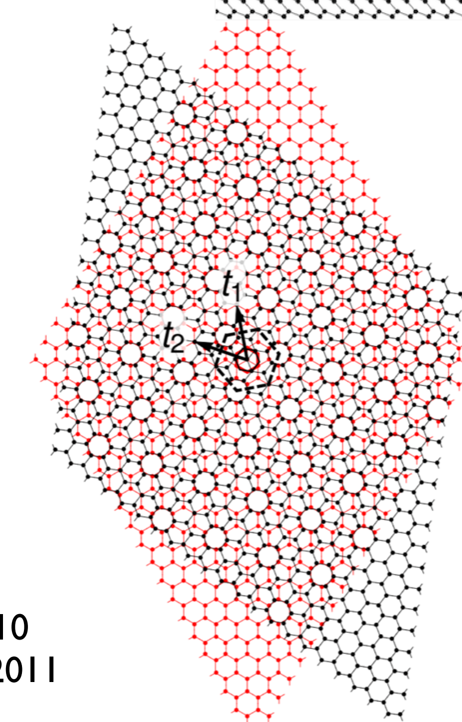


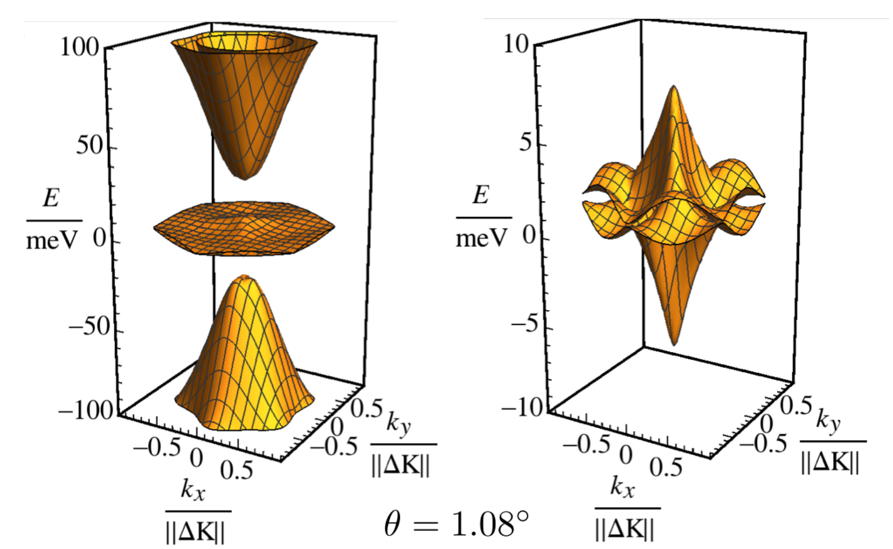
- Twisted bilayer graphene

$$\Omega_{\rm FB} = \Omega_{\rm BZ}^{\rm superlattice} \sim \frac{\Omega_{\rm BZ}^{\rm graphene}}{N}$$

$N \sim 10^4$ at $\theta \sim 1.1°$

Suarez-Morell et al. 2010
Bistritzer-MacDonald 2011

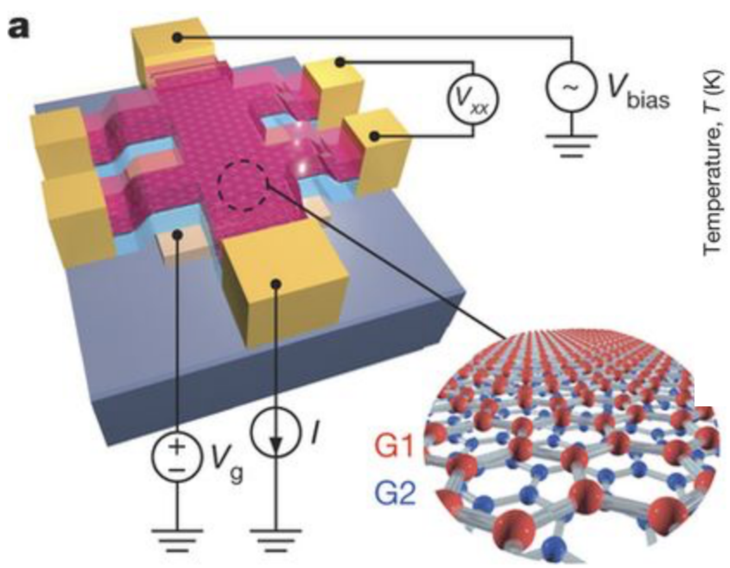


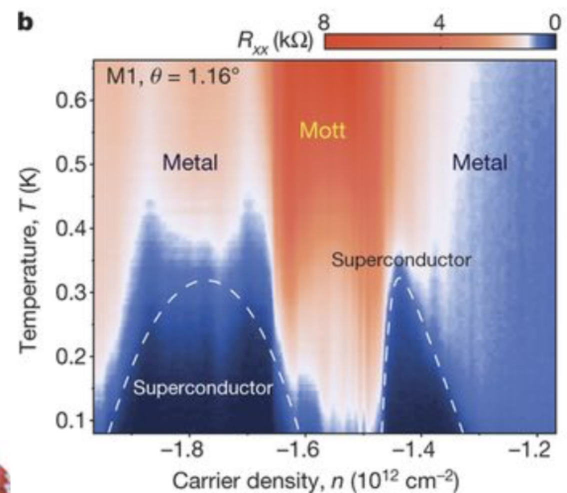


Unconventional superconductivity in magic-angle graphene superlattices

Yuan Cao, Valla Fatemi, Shiang Fang, Kenji Watanabe, Takashi Taniguchi, Efthimios Kaxiras & Pablo Jarillo-Herrero

Nature **556**, 43–50 (05 April 2018) | Download Citation



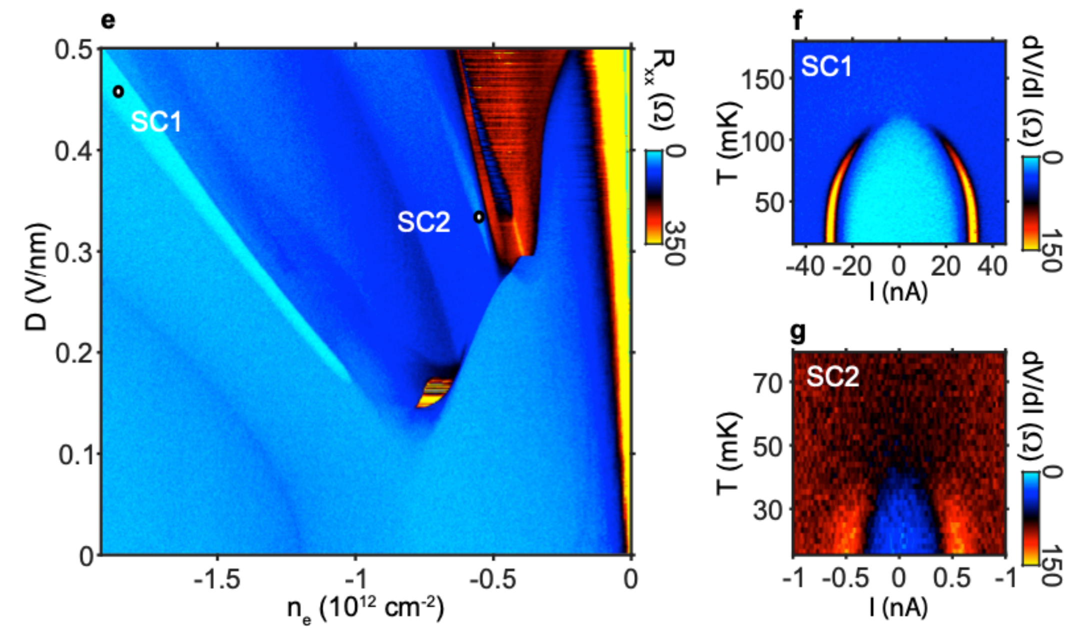


Article | Published: 01 September 2021

Superconductivity in rhombohedral trilayer graphene

Haoxin Zhou, Tian Xie, Takashi Taniguchi, Kenji Watanabe & Andrea F. Young

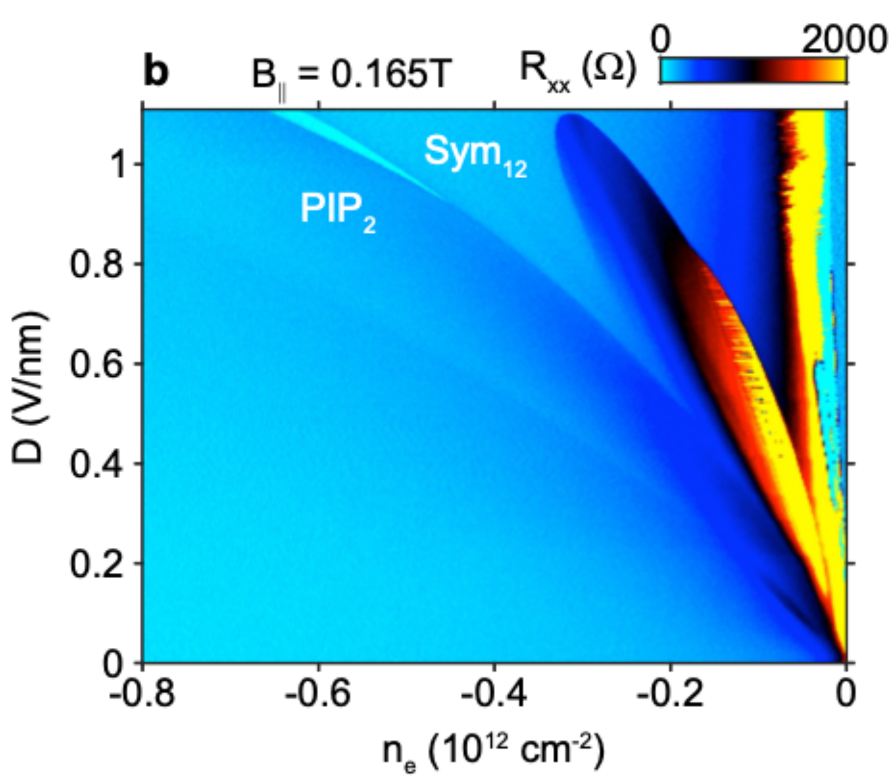
Nature **598**, 434–438 (2021) | Cite this article



Isospin magnetism and spin-polarized superconductivity in Bernal bilayer graphene

HAOXIN ZHOU, LUDWIG HOLLEIS, YU SAITO, LIAM COHEN, WILLIAM HUYNH, CAITLIN L. PATTERSON, FANGYUAN YANG, TAKASHI TANIGUCHI, KENJI WATANABE, [...] ANDREA F. YOUNG +1 authors Authors Info & Affiliations

SCIENCE • 13 Jan 2022 • Vol 375, Issue 6582 • pp. 774-778 • DOI: 10.1126/science.abm8386

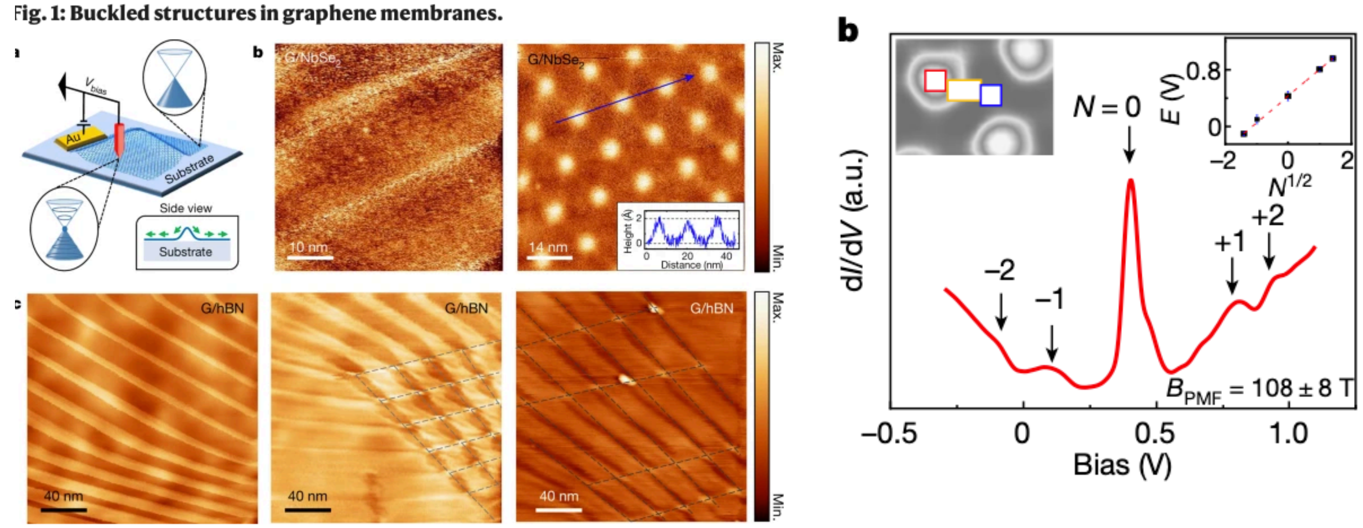


Evidence of flat bands and correlated states in buckled graphene superlattices

Jinhai Mao, Slaviša P. Milovanović, Miša Anđelković, Xinyuan Lai, Yang Cao, Kenji Watanabe, Takashi Taniguchi, Lucian Covaci, Francois M. Peeters, Andre K. Geim, Yuhang Jiang & Eva Y. Andrei

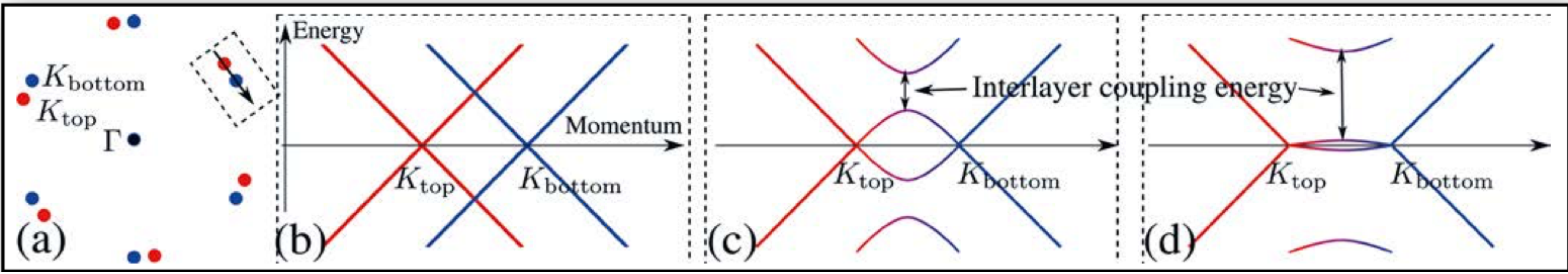
Nature **584**, 215–220(2020) | Cite this article

Fig. 1: Buckled structures in graphene membranes.



Signature of correlations, not superconductivity

Twisted bilayer graphene



Possible to extend to twisted double bilayers etc.

TTH & T. Hyart, Europhysics News 50, 24 (2019)

But wait!

If Fermi speed tends to zero, electrons become completely localised!

How can they superconduct?

Superfluid weight D_s in $\mathbf{j}_s = D_s \nabla \varphi$

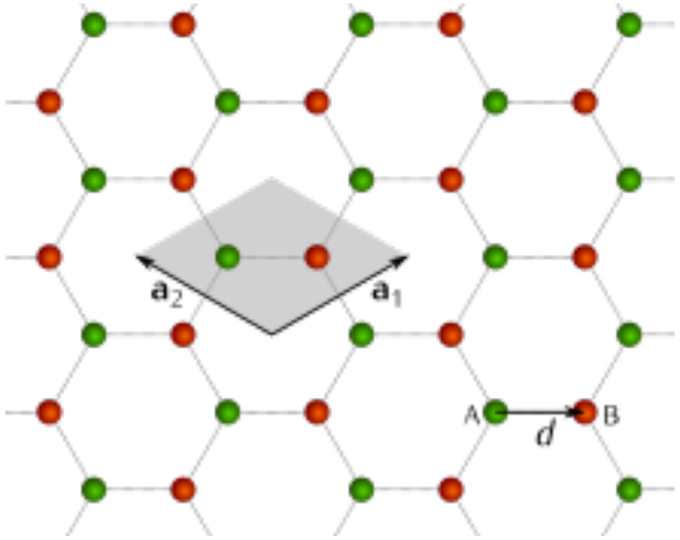
Conventional superconductor: $D_s = \frac{n_s}{m_{\mathrm{eff}}}$

2D: transition at $T = T_{\mathrm{BKT}}$, $T_{\mathrm{BKT}} = \frac{\pi}{8}\sqrt{\det D^s(T_{\mathrm{BKT}})}$

BKT=Berezinskii-Kosterlitz-Thouless

For a trivial flat band, D_s=0 and T_{BKT}=0

Quantum geometry of the Bloch bands may change the superfluid weight in multiorbital systems



Multiorbital superconductor:

Bloch functions are vectors $|\psi_{\mathbf{k}}\rangle = \begin{pmatrix} \chi_{\mathbf{k},A} \\ \chi_{\mathbf{k},B} \end{pmatrix}$

$$D_s = D_{s,\mathrm{conv}} + D_{s,\mathrm{geom}}$$

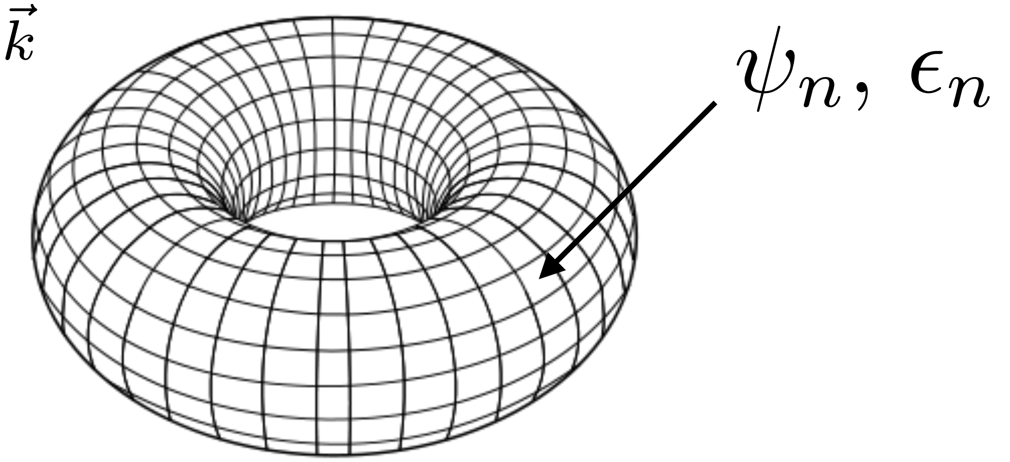
$\frac{n_s}{m_{\mathrm{eff}}}$

"geometric contribution"

S. Peotta & P. Törmä, Nat. Commun. **6**, 8944 (2015)
Törmä, Peotta, Bernevig, Nat. Rev. Phys. (2022)
K.E. Huhtinen, *et al.*, PRB **106**, 014518 (2022)
… and many others

Bloch's theorem

- Position representation of the Hamiltonian: $H(\vec{r}) = \langle \vec{r} | \hat{H} | \vec{r} \rangle$
- Lattice periodic Hamiltonian:
  $H(\vec{r} + \vec{n} \cdot \vec{a}) = H(\vec{r}), \quad \vec{n} \in \mathbb{Z}^d, \vec{a} \in \mathbb{R}^d$
- Define $H(\vec{k}) = \langle \vec{k} | \hat{H} | \vec{k} \rangle$
- Lattice periodicity requires that $H(\vec{k} + \vec{n} \cdot \vec{b}) = H(\vec{k})$, where $\vec{b}_i \cdot \vec{a}_j = 2\pi\delta_{ij}$.
- First Brillouin zone: $\{\vec{k}\}_{\mathrm{1stBZ}} = \{\vec{k} \mod \vec{b}\} \cong T^d$
- $H(\vec{k})$: $T^d \mapsto$ eigenvectors $|\epsilon_{n\vec{k}}\rangle$ and -values $\epsilon_{n\vec{k}}$



Berry phase

Cyclic adiabatic time evolution:
$H = H(X(t))$, instantaneous eigenstate $|\phi_n(X(t))\rangle$
State at time t:

$$|\Psi_n(t)\rangle = e^{i\gamma_n(t)} e^{-\frac{i}{\hbar}\int_0^t dt' \epsilon_n(X(t'))} |\phi_n(X(t))\rangle$$

geometric phase

dynamic phase

Geometric phase

Adiabatic changes of a wf with respect to a change in parameter X:
Geometric phase

$$\gamma = \int_{C_X} \underbrace{-i\langle\psi|\nabla_X\psi\rangle}_{\vec{A}} \cdot d\vec{X}$$

Berry connection,
analogue of vector potential

Closed loop: *Berry phase*

$$\gamma_X = \oint_C \vec{A}\cdot d\vec{X} = \int_{S_C} \underbrace{\nabla\times\vec{A}}_{=\,\Omega,\ \text{berry curvature}} dS$$

Chern number: $n = \frac{-i}{2\pi}\int_{\text{closed surface}} \Omega dS \in \mathbb{Z}$

"Topological charge"

Quantum geometric tensor

$$\frac{\partial}{\partial p_i}$$

$$B_{\alpha,ij}(\mathbf{p}) = \langle \partial_i \psi_\alpha | (1_N - |\psi_\alpha\rangle\langle\psi_\alpha|) | \partial_j \psi_\alpha \rangle$$
$$= \mathrm{Tr}[(\partial_i P_\alpha)(1 - P_\alpha)(\partial_j P_\alpha)]$$

$$P_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$B_{\alpha,ij}(\mathbf{p}) \equiv g_{\alpha,ij}(\mathbf{p}) - \frac{i}{2}\Omega_{\alpha,ij}(\mathbf{p})$$

Quantum metric
“Fubini-Study metric”

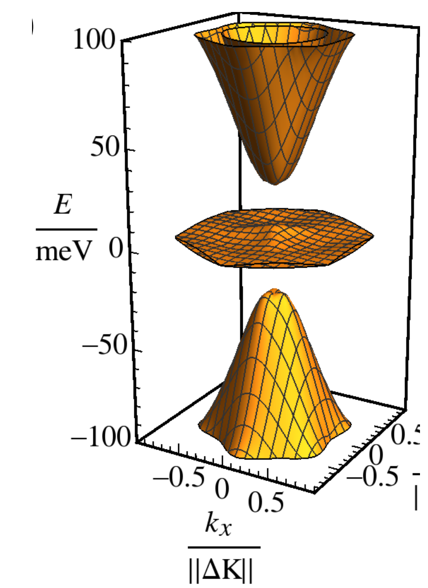
Berry curvature

Pedagogical introduction:
A. Graf & F. Piéchon, PRB **104**, 085114 (2021)

For isolated flat bands, the quantum metric provides the superfluid weight

$$[D_s]_{ij} = \frac{4f(1-f)}{(2\pi)^{D-1}}|U|n_\phi \mathcal{M}_{ij},$$

$$\mathcal{M}_{ij} = \frac{1}{2\pi}\int_{\mathrm{B.Z.}} d^D k\, \mathrm{Re}[\mathcal{B}_{ij}(\boldsymbol{k})]$$



Note: one needs to consider the smallest possible quantum metric

S. Peotta & P. Törmä, Nat. Commun. **6**, 8944 (2015)
Törmä, Peotta, Bernevig, Nat. Rev. Phys. (2022)
K.E. Huhtinen, *et al.*, PRB **106**, 014518 (2022)
… and many others

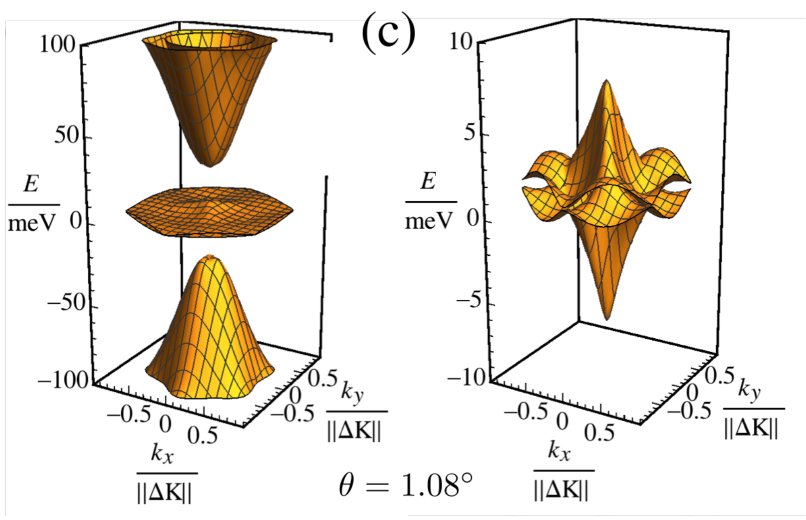
Approximate flat band

Flat band picture is an idealisation.
Typically: Δ vs linewidth δE. Use $\Delta_{\mathrm{FB}} \equiv g\Omega_{\mathrm{FB}}/(8\pi)$

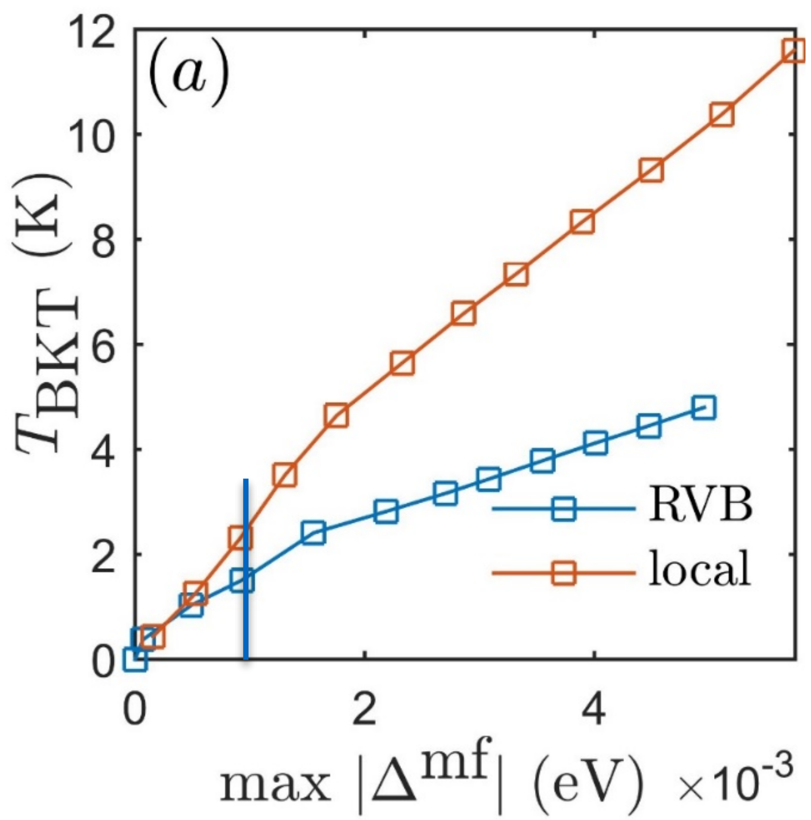
- If $\Delta_{\mathrm{FB}} > \delta E$, $\Delta \approx \Delta_{\mathrm{FB}}$
- For $\Delta_{\mathrm{FB}} \lesssim \delta E$, $\Delta \approx \delta E e^{-1/(g\nu_F)}$ or something more complicated…

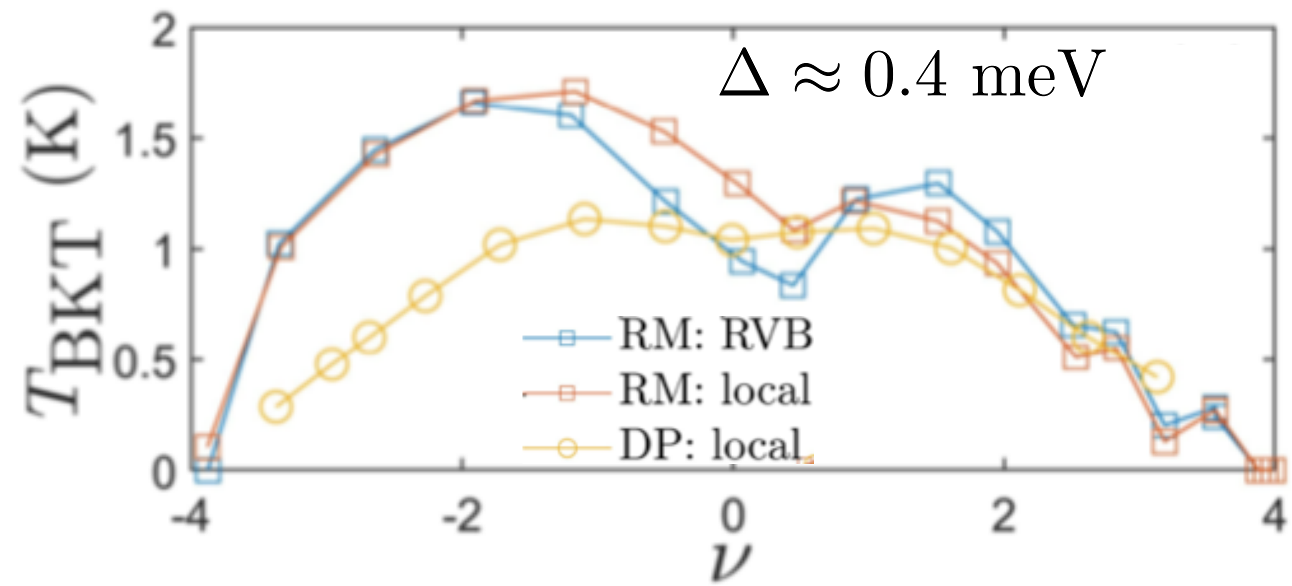
Alternative (often takes place roughly simultaneously):

- Conventional: $D_{s,\mathrm{conv.}} > D_{s,\mathrm{geom}}$
- Flat-band limit: $D_{s,\mathrm{conv.}} < D_{s,\mathrm{geom}}$

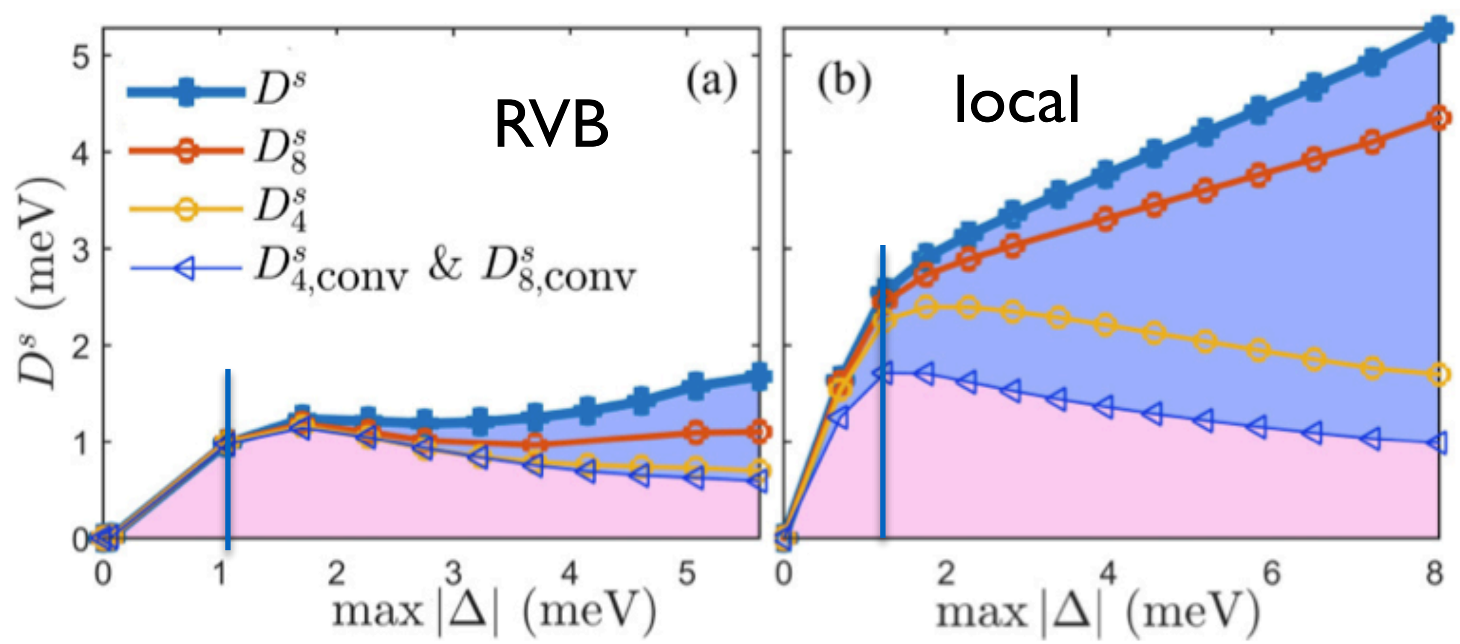


BKT physics in twisted bilayer graphene





Conventional and
geometric contributions



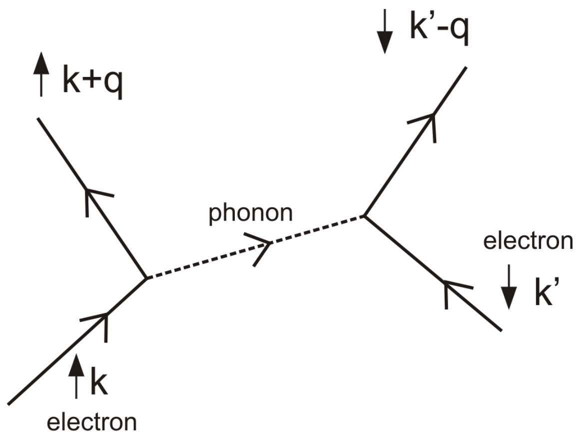
A. Julku, T. Peltonen, L. Liang, TTH, P. Törmä, Phys. Rev. B **101**, 060505 (2020)
(see also X. Hu, *et al.*, PRL **123**, 237002 (2019); Xie, *et al.,* PRL **124**, 167002 (2020), Peri, *et al.*, arXiv:2008.02288).

Electron-phonon model

Previous models independent of the source of attractive interaction

Eliashberg theory:

- (Retarded) attractive interaction from electron-phonon coupling
- Possibility of including direct Coulomb interactions (Anderson-Morel pseudopotential)



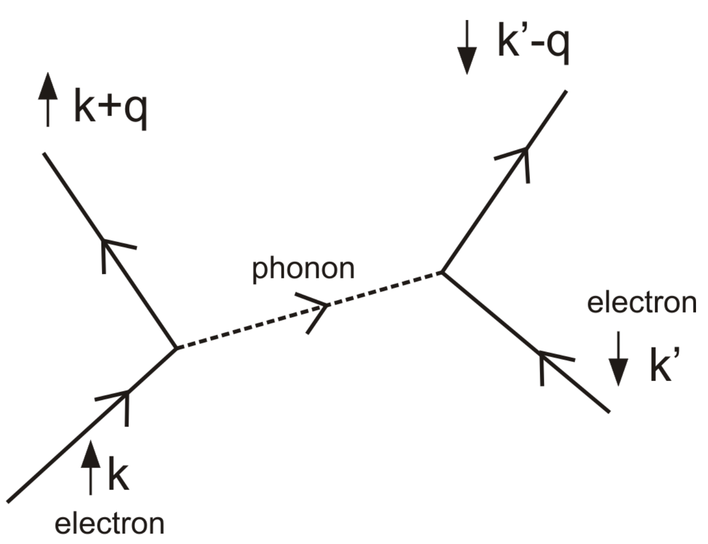
Screened Coulomb; including retardation: coupling constant u, electron-phonon: coupling constant λ

$$\Rightarrow \text{effective BCS interaction } g = \lambda - u$$

Mental picture

Attractive interaction from exchange of a boson with frequency ω_b

Interaction energy: $\lambda \sim \frac{g^2}{\omega_b}$

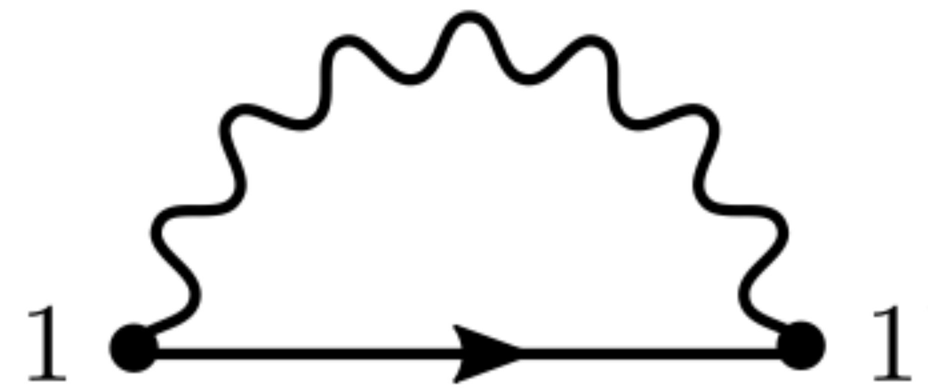


Instead of phonons, the boson could also be e.g., magnon

See R. Ojajärvi's doctoral dissertation (Univ. Jyväskylä 2022)

Eliashberg theory

Electron-phonon self-energy Σ
in the presence of superconductivity



$$\hat{\Sigma}_{\mathrm{e-ph}}(p) = i[1 - Z(p)]\omega_n \tau_3 + \chi(p)\tau_0 + i\hat{\phi}(p)$$

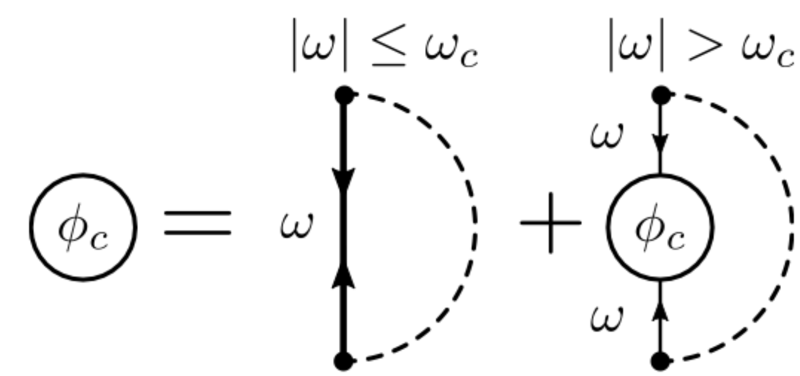
$$\hat{\Delta}(\omega_n) \equiv \frac{\hat{\phi}(\omega_n)}{Z(\omega_n)}$$

Vector in Matsubara frequencies ω_n!

Coupled self-consistency equations for ϕ, Z, χ

Direct Coulomb: non-retarded interaction

$$\phi(\omega_n) = \phi_{\mathrm{ph}}(\omega_n) + \phi_{\mathrm{C}}$$



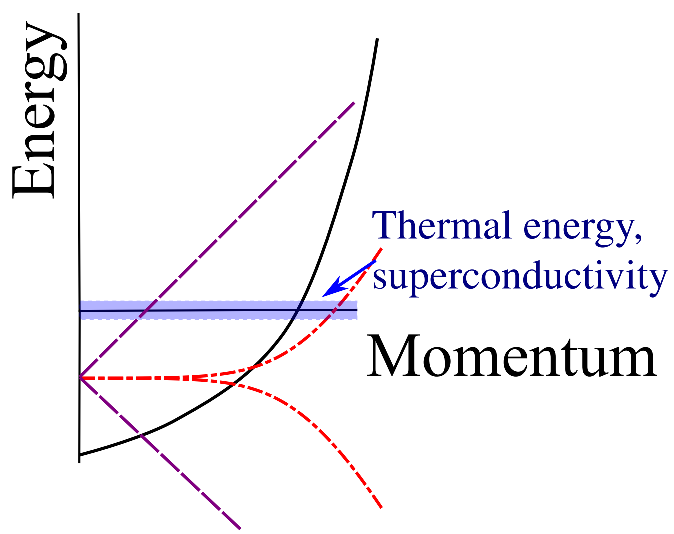
$\Rightarrow$ competition between attraction (ϕ_{ph}) and repulsion (ϕ_{C})

$\Rightarrow$ solution with the Anderson-Morel pseudopotential $\quad U^{+} = U/(1 + U\chi_{\mathrm{h}})$

Electron-phonon model (Eliashberg)

(Mean field, qualitative ideas partially valid also for other interacting states)

Fermi surface:

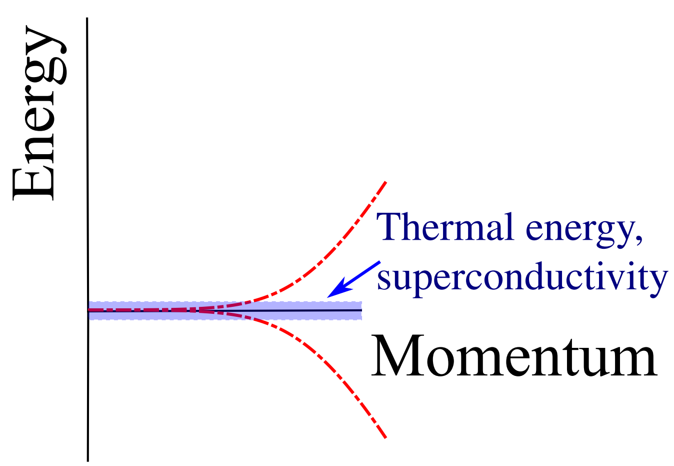


$$T_c = \frac{\hbar\omega_D}{1.45 k_B} \exp\left[-\frac{1.04(1+\lambda)}{\lambda - \mu_c^*(1+0.62\lambda)}\right] \sim \hbar\omega_D \exp(-1/\lambda)$$

McMillan, PRB (1968)
Allen & Dynes, PRB (1975)

$$\lambda = g\nu(\epsilon_f)$$

Flat band:



Eliashberg + Coulomb:

$$T_C \propto (\lambda - u^*)$$

Ojajärvi, Hyart, Silaev & TTH, PRB 2018

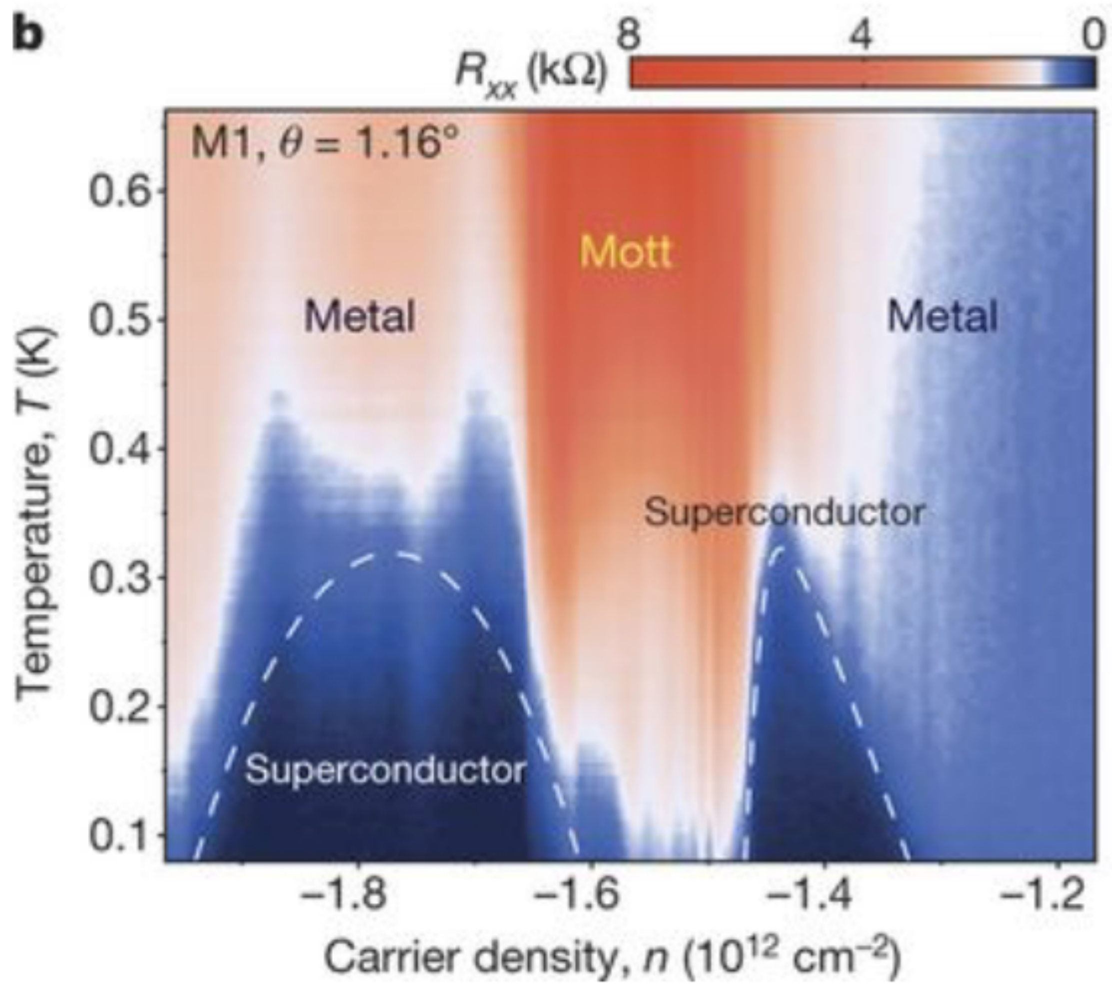
$$\lambda = \frac{g^2}{\omega_E^2}\frac{\Omega_{\mathrm{FB}}}{\Omega_{\mathrm{BZ}}}, \quad u = \frac{U\Omega_{\mathrm{FB}}}{\Omega_{\mathrm{BZ}}}$$

$$u^* = \frac{u}{1 + u\chi_h}$$

Other bands!

Other correlated states?

If the BCS interaction $\lambda - \frac{u}{1+u\chi_h} < 0$, repulsive interactions dominate

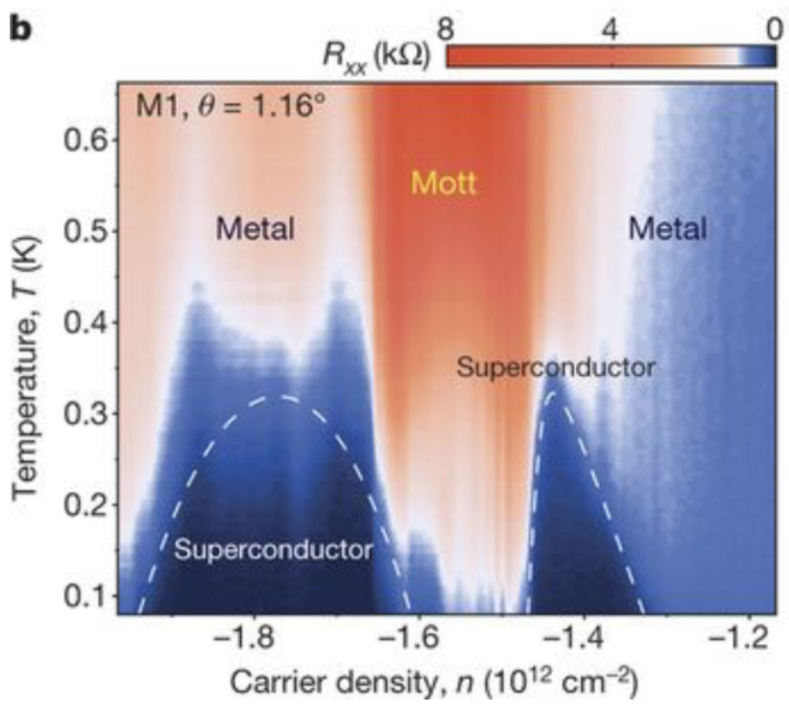


Cao, et al., 2018

May depend on doping?

Other states? Doping dependence?

Insulator state between? Magnetic?

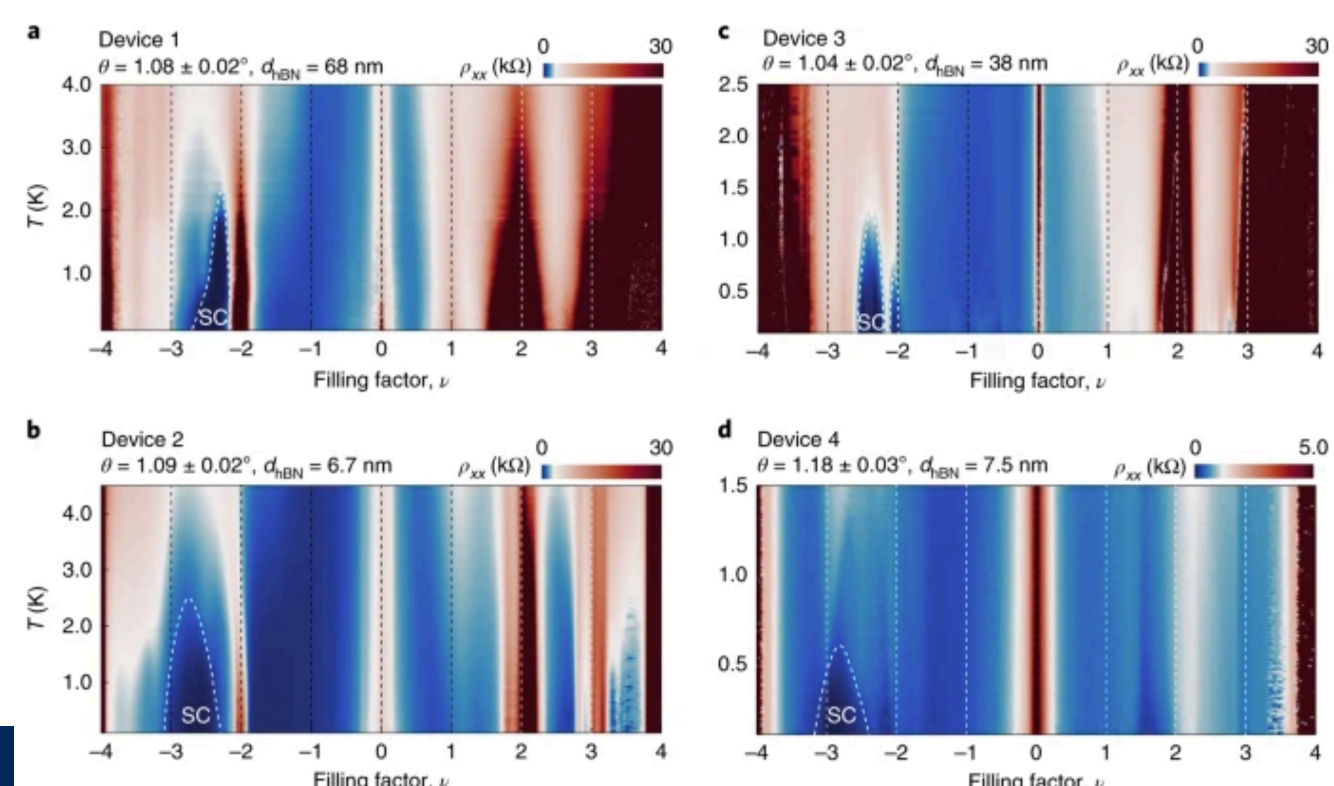


Independent superconductors and correlated insulators in twisted bilayer graphene

Yu Saito, Jingyuan Ge, Kenji Watanabe, Takashi Taniguchi & Andrea F. Young

Nature Physics **16**, 926–930(2020) | Cite this article

Here: vary the distance to a nearby gate
$\Rightarrow$ screening of Coulomb interactions is affected
$\Rightarrow$ changes the insulating phase, not the SC phase!
$\Rightarrow$ independent mechanisms behind the two!



We do not yet know the origin of superconductivity in the few-layer graphenes

TBG experiments can be explained if $\lambda_{\text{eff}} \sim 50\ \text{meV}{\cdot}(\text{nm})^2$

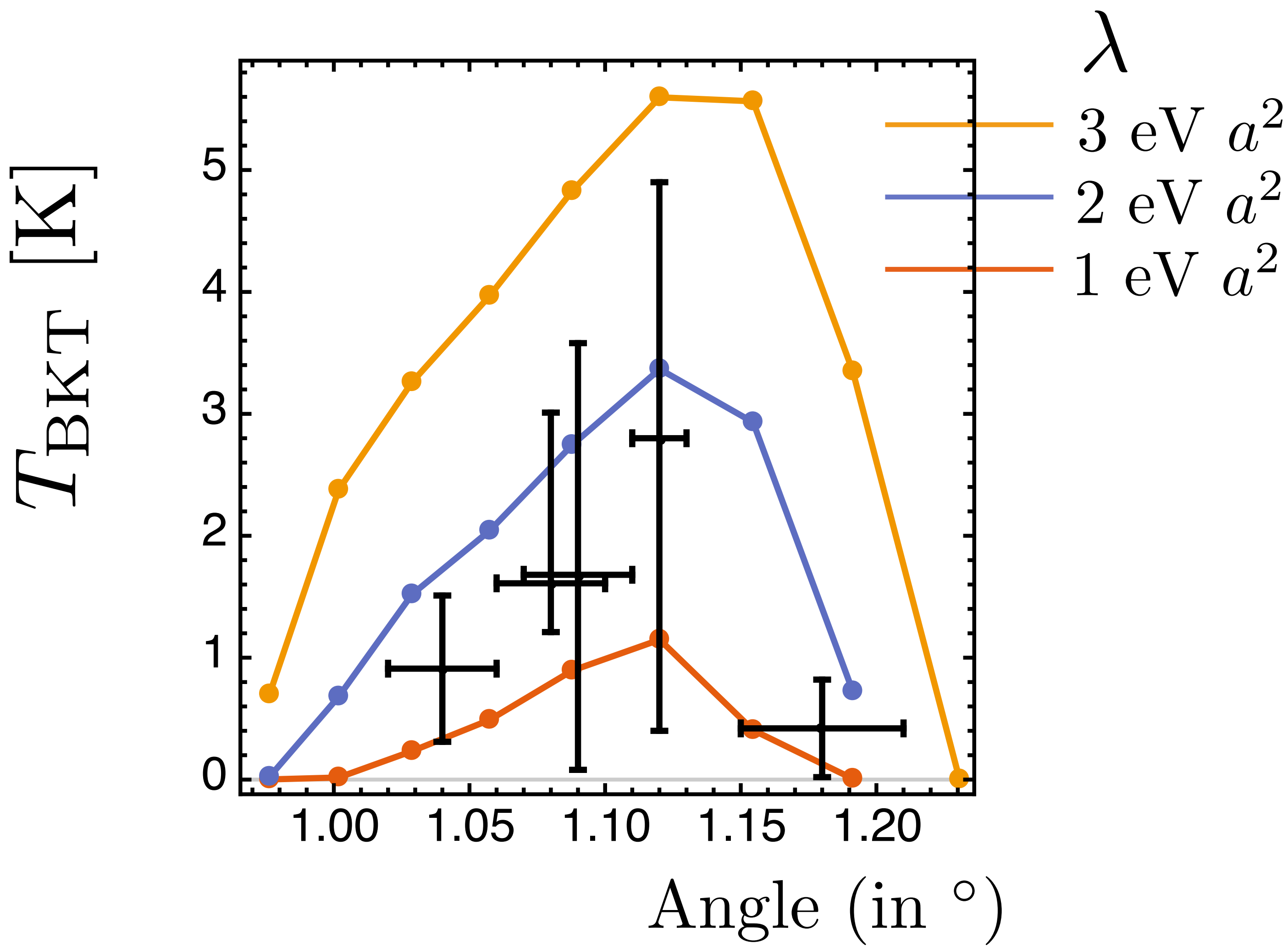
Wu, das Sarma (several works): $\lambda_{\text{eph}}^{\text{bare}} \sim 150 \ldots 200\ \text{eV} \cdot (\text{nm})^2$

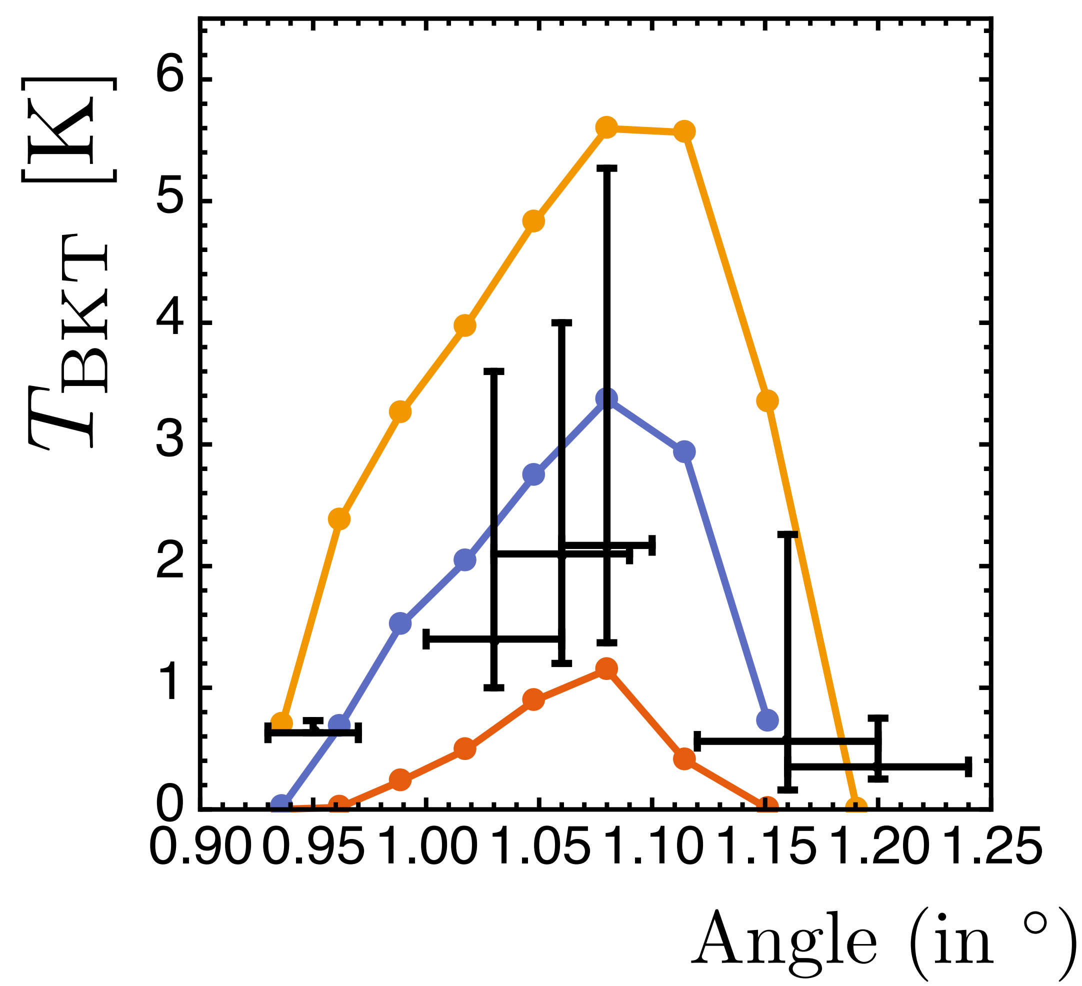
BUT: those values would require $g_{\text{eph}} > \omega_D$

There is still uncertainty of the mechanism - some authors claim it must be an interaction driven by higher order theory from direct Coulomb interactions

Indications: nematic superconductivity, spin triplet superconductivity

Comparison to experiments



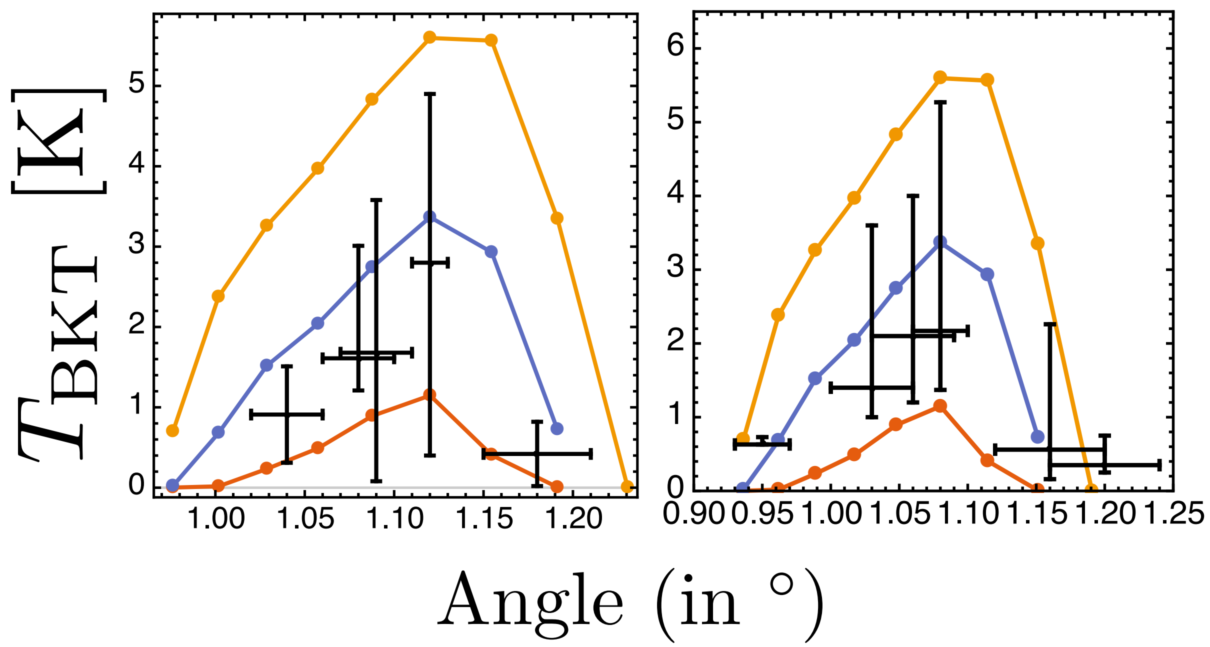


Data from Saito, et al., Nat Phys. (2020) (A. Young's group), angles shifted by 0.07°, $T_{\mathrm{BKT}} = 0.7 T_c$

Data from Cao, et al., arXiv:2004.04148 (Jarillo-Herrero group), angles shifted by 0.03°, $T_{\mathrm{BKT}} = 0.7 T_c$

Caveats: see discussion in Teemu Peltonen's thesis, https://jyx.jyu.fi/handle/123456789/71509

Experiments



Best fits with $\lambda \sim 2$ eVa^2

Wu, MacDonald, Martin, PRL 2018: $\lambda < 2.7$ eV a^2

$\Rightarrow \Delta_{\mathrm{max}} \sim 2$ meV

Mean-field $T_c^{\mathrm{mf}} \sim 6$ K

Electron-phonon mechanism with roughly similar λ seems to be consistent with

- trilayer ABC graphene T_c (Chou, et al. PRL 2021)
- Bernal bilayer T_c (Chou, et al., arXiv:2110.12303)

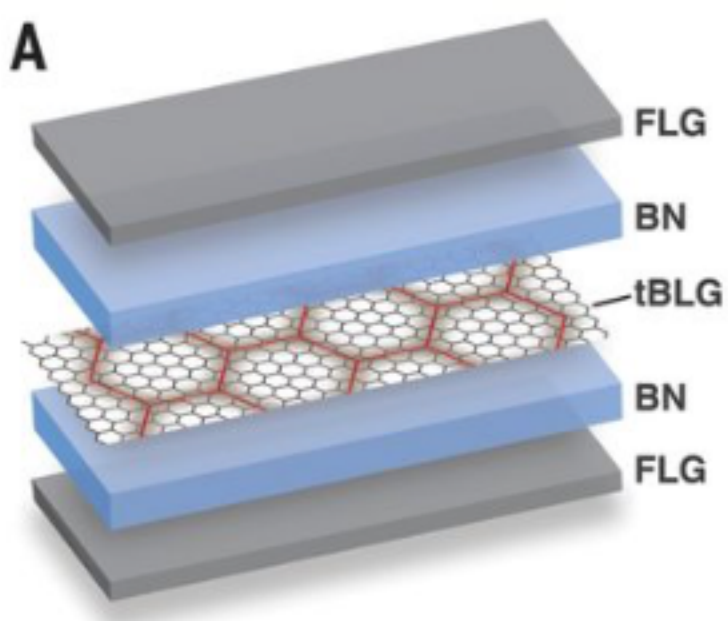
(those are **not** in the flat-band regime)

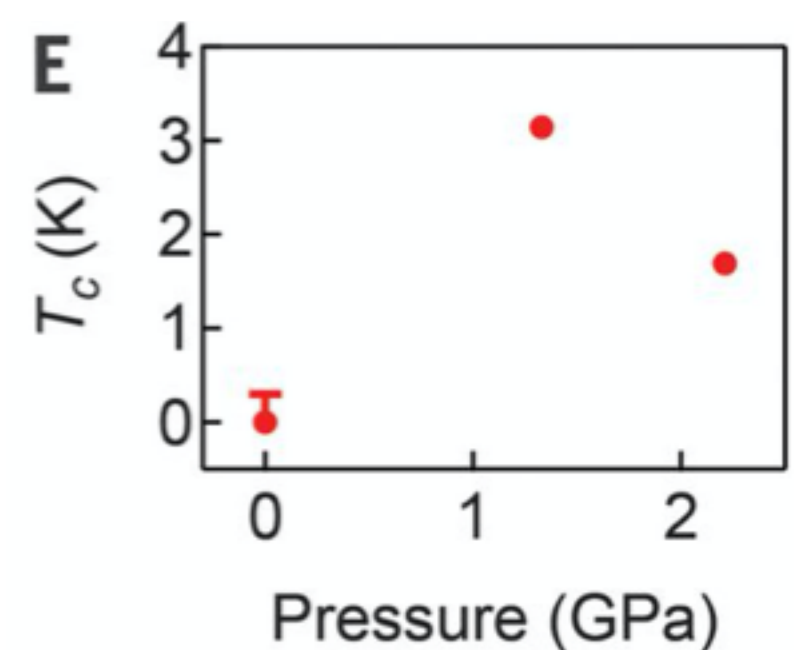
TBG superconductivity from flat-band perspective?

- Flat-band superconductivity: $\Delta_{\mathrm{FB}} \sim \frac{g\Omega_{\mathrm{FB}}}{8\pi} \gtrsim \delta E_{\mathrm{band}}$
- TBG: $\Omega_{\mathrm{FB}} = \Omega_{\mathrm{BZ}}^{\mathrm{superlattice}} \sim \frac{1}{a^2 N} \Rightarrow T_{\mathrm{BKT}}, T_c^{\mathrm{mf}} \sim$ few K

  $N \sim 10^4$ at $\theta \sim 1.1°$
- To increase them, look for systems with a larger Ω_{BZ}!
- Example: TBG with pressure $\Rightarrow$ larger $\theta^{\mathrm{magic}} \Rightarrow$ larger T_c (?)

Yankowitz, et al. Science 2019





Coherence length

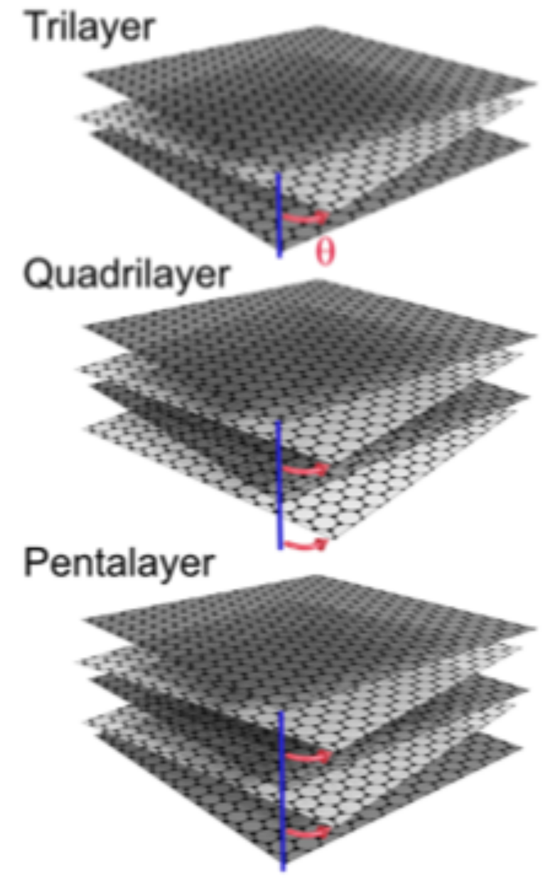
- BCS: $\xi_0 = \frac{\Delta}{\hbar v_F}$.
- Flat band: $v_F \to 0$! However, replace $\xi_0 \sim k_{\mathrm{FB}}^{-1} \sim \Omega_{\mathrm{FB}}^{-1/d}$

Kopnin & TTH, arXiv:1210.7075

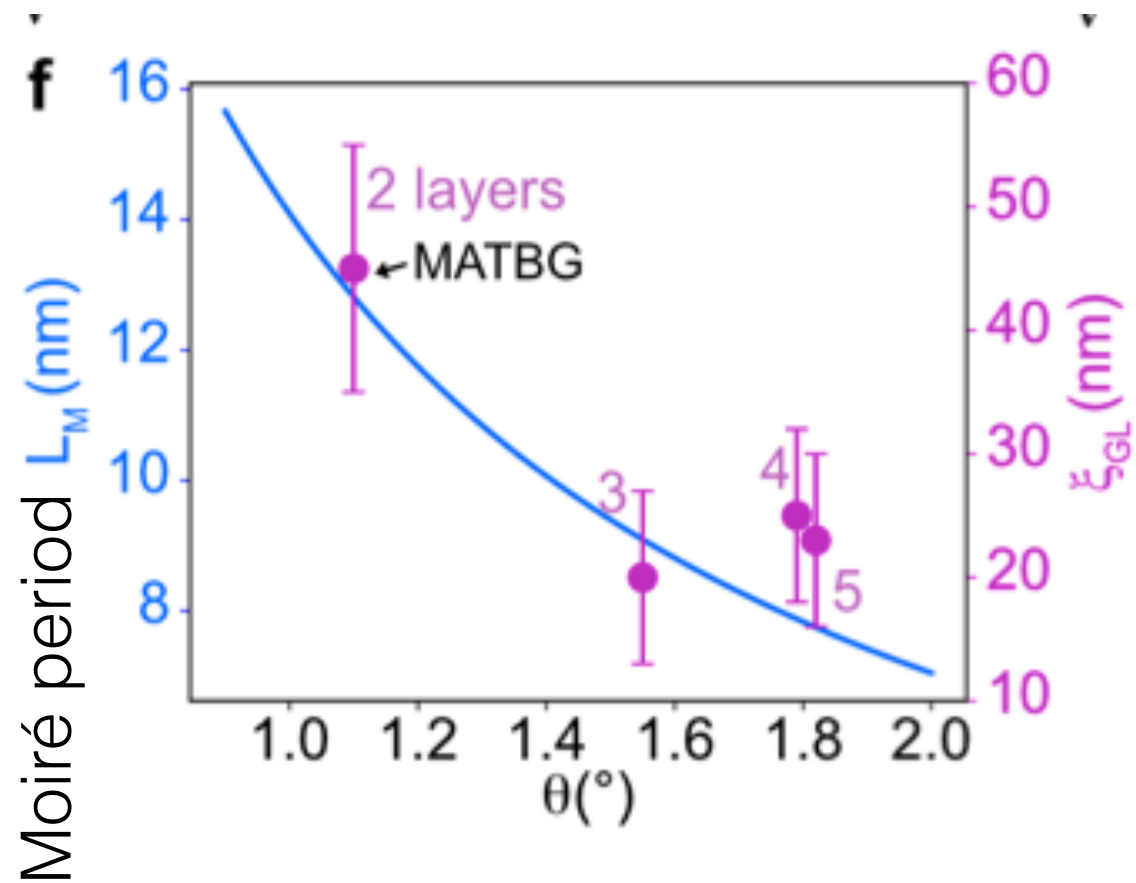
Ascendance of Superconductivity in Magic-Angle Graphene Multilayers

Yiran Zhang[1,2,3*], Robert Polski[1,2*], Cyprian Lewandowski[2,3], Alex Thomson[2,3,4], Yang Peng[5], Youngjoon Choi[1,2,3], Hyunjin Kim[1,2,3], Kenji Watanabe[6], Takashi Taniguchi[6], Jason Alicea[2,3], Felix von Oppen[7], Gil Refael[2,3], and Stevan Nadj-Perge[1,2†]

arXiv:2112.09270



Magic angle increases with increasing # of layers



$$\Omega_{\rm FB} \sim 1/L_m^2$$

Consistent with flat-band superconductivity!

$$T_c/T_{c0} = 1 - (2\pi\xi_{\rm GL}^2/\Phi_0)B_\perp$$

Is superconductivity in graphene-based systems unconventional?

Pauli-limit violation and re-entrant superconductivity in moiré graphene

Yuan Cao, Jeong Min Park, Kenji Watanabe, Takashi Taniguchi & Pablo Jarillo-Herrero

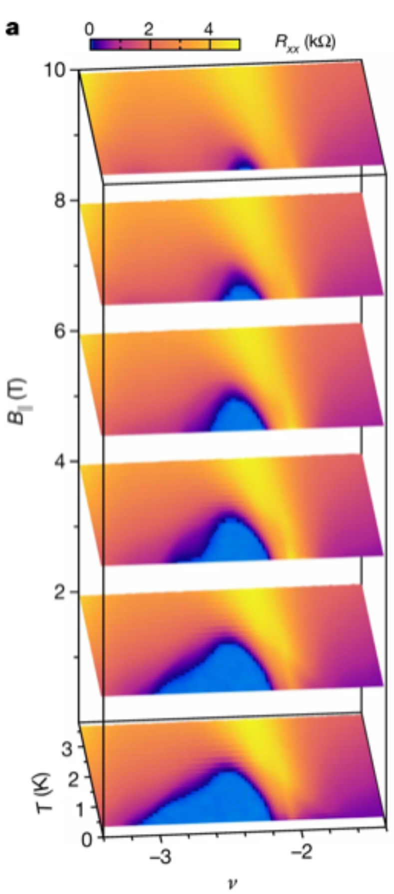
Nature **595**, 526–531 (2021) | Cite this article

Nematicity and competing orders in superconducting magic-angle graphene

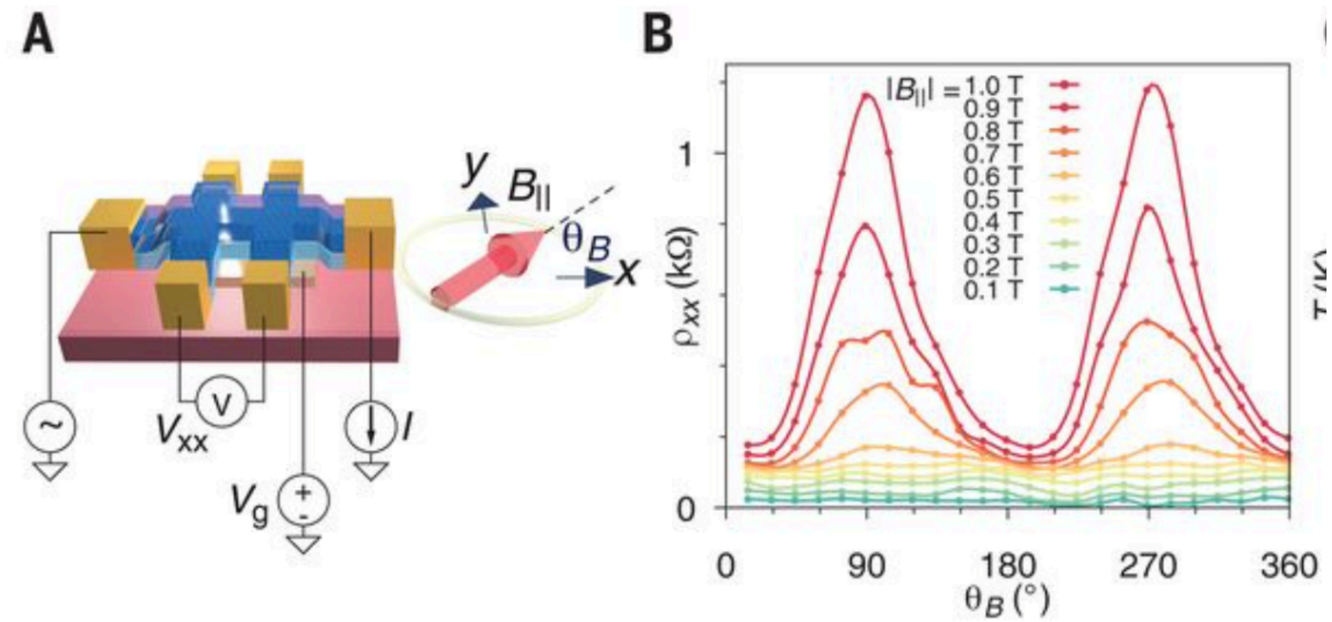
YUAN CAO, DANIEL RODAN-LEGRAIN, JEONG MIN PARK, NOAH F. Q. YUAN, KENJI WATANABE, TAKASHI TANIGUCHI, RAFAEL M. FERNANDES, LIANG FU, AND PABLO JARILLO-HERRERO Authors Info & Affiliations

Fig. 2: Large Pauli-limit violation in MATTG.

From: Pauli-limit violation and re-entrant superconductivity in moiré graphene



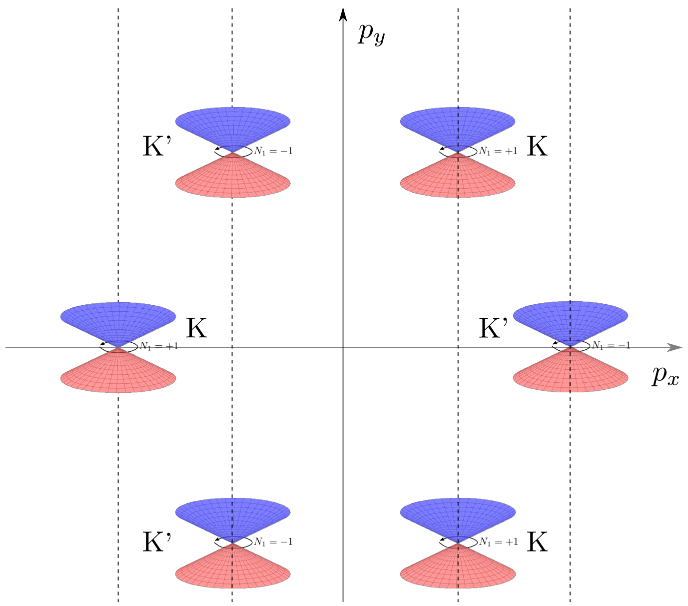
Pauli limit: $h_c \sim \Delta$
(Fermi systems: $h_c = \Delta/\sqrt{2}$)



Anisotropic magnetoresistance
(Breaks $C(6)$ symmetry)

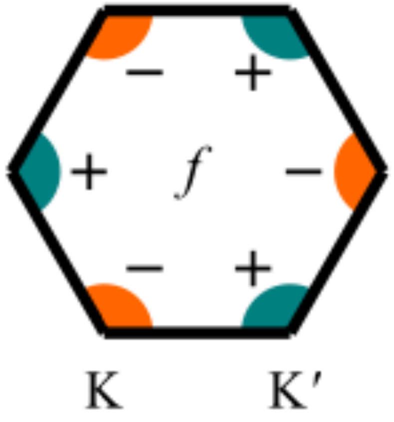
Pairing symmetry

All graphene models: pairs of flat bands around valleys K/K'



$\Rightarrow$ degeneracy between valley-odd and valley-even Δ!
$\Rightarrow$ valley odd consistent with even-spin (spin triplet) pairing!

f-wave superconductivity!



See discussion: Chou, Wu, Sau, Das Sarma, PRL 2021

May explain the recently observed large in-plane critical fields (beyond Pauli limit)

Conclusions

- Increasing ν_F: instabilities to interaction-driven correlated states more likely, e.g., superconductivity
- Ultimate limit: flat-band superconductivity $\Delta \gtrsim \delta E \Rightarrow T_c \sim \Delta \approx \Delta_{\mathrm{FB}} = g\Omega_{\mathrm{FB}}$
- Independent of pairing mechanism, but may be used to explain graphene-based superconducting states using electron-phonon mediated interaction
- Valley degeneracy $\Rightarrow$ possibility for spin triplet pairing and breakdown of the Clogston limit
- Increasing T_c: look for systems with larger Ω_{FB}.

For example: topological insulator with surface polarisation:
J. Nissinen, TTH and G.E. Volovik, PRB **103,** 245115 (2021)